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BY

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WILLIAM F. BRADBURY, A. M.,

HOPKINS MASTER IN THE CAMBRIDGE HIGH SCHOOL; AUTHOR OF A TREATISE ON TRIGONOMETRY AND SURVEYING, AND OF AN ELEMENTARY ALGEBRA.

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PREFACE.

A LARGE number of the Theorems usually presented in textbooks of Geometry are unimportant in themselves and in no way connected with the subsequent Propositions. By spending too much time on things of little importance, the pupil is frequently unable to advance to those of the highest practical value. In this work, although no important Theorem has been omitted, not one has been introduced that is not necessary to the demonstration of the last Theorem of the five Books, namely, that in relation to the volume of a sphere. Thus the whole constitutes a single Theorem, without an unnecessary link in the chain of reasoning.

These five Books, including Ratio and Proportion, are presented in eighty-one Propositions, covering only seventy pages. This brevity has been attained by omitting all unconnected propositions, and adopting those definitions and demonstrations that lead by the shortest path to the desired end. At the close of each Book are Practical Questions, serving partly as a review, partly as practical applications of the principles of the Book, and partly as suggestions to the teacher. As those who have not had experience in discovering methods of demonstration have but little real acquaintance with Geometry, there have been added to each Book, for those who have the time and the ability, Theorems for original demonstration. These Exercises, with different methods of proving propositions already demonstrated, include those that are usually inserted, but whose demonstration in this work has been omitted. In some of these Exercises references are given to the necessary propositions; in some suggestions are made; and in a few cases the figure is constructed as the proof will require.

A sixth Book of Problems of Construction is added, which is followed by Problems for the pupil to solve. This Book, or any part of it, if thought best, can be taken immediately after completing Book III.

W. F. B.

CAMBRIDGE, MASS., April, 1872.

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INTRODUCTORY DEFINITIONS.

1. **Mathematics** is the science of quantity.

2. **Quantity** is that which can be measured; as distance, time, weight.

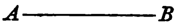
3. **Geometry** is that branch of mathematics which treats of the properties of extension.

4. **Extension** has one or more of the three dimensions, length, breadth, or thickness.

5. A **Point** has position, but not magnitude.

6. A **Line** has length, without breadth or thickness.

7 A **Straight Line** is one whose direction is the same throughout; as *A B*.

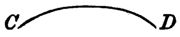


A straight line has two directions exactly opposite, of which either may be assumed as its direction.

The word *line*, used alone in this book, means a straight line.

8. *Corollary.* Two points of a line determine its position.

9. A **Curved Line** is one whose direction is constantly changing; as *C D*.



10. A **Surface** has length and breadth, but no thickness.

11. A **Plane** is such a surface that a straight line joining any two of its points is wholly in the surface.

12. A **Solid** has length, breadth, and thickness.

13. *Scholium.* The boundaries of solids are **surfaces**; of surfaces, lines; the ends of lines are points.

14. A **Theorem** is something to be proved.

15. A **Problem** is something to be done.

16. A **Proposition** is either a theorem or a problem.

17. A **Corollary** is an inference from a proposition or statement.

18. A **Scholium** is a remark appended to a proposition.

19. An **Hypothesis** is a supposition in the statement of a proposition, or in the course of a demonstration.

20. An **Axiom** is a self-evident truth.

AXIOMS.

1. If equals are added to equals, the sums are equal.
2. If equals are subtracted from equals, the remainders are equal.
3. If equals are multiplied by equals, the products are equal.
4. If equals are divided by equals, the quotients are equal.
5. Like powers and like roots of equals are equal.
6. The whole of a magnitude is greater than any of its parts.
7. The whole of a magnitude is equal to the sum of all its parts.
8. Magnitudes respectively equal to the same magnitude are equal to each other.
9. A straight line is the shortest distance between two points.

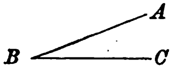
BOOK I.

ANGLES, LINES, POLYGONS.

ANGLES.

DEFINITIONS.

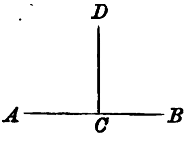
1. An **Angle** is the difference in direction of two lines.



If the lines meet, the point of meeting, B, is called the *vertex;* and the lines $A B$, $B C$, the *sides* of the angle.

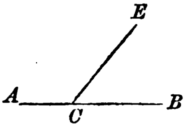
If there is but one angle, it can be designated by the letter at its vertex, as the angle B; but when a number of angles have the same vertex, each angle is designated by three letters, the middle letter showing the vertex, and the other two with the middle letter the sides; as the angle $A B C$.

2. If a straight line meets or crosses another so as to make the angles equal, each of these angles is a *right angle;* and the two lines are perpendicular to each other. Thus, $A C D$ and $D C B$, being equal, are right angles, and $A B$ and $D C$ are perpendicular to each other.



3. An **Acute Angle** is less than a right angle; as $E C B$.

4. An **Obtuse Angle** is greater than a right angle; as $A C E$.



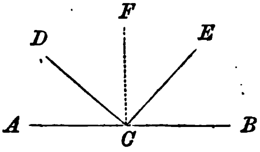
Acute and obtuse angles are called oblique angles.

5. The **Complement** of an angle is a right angle minus the given angle. Thus (Fig. in Art. 7), the complement of ACD is $ACF - ACD = DCF$.

6. The **Supplement** of an angle is two right angles minus the given angle. Thus (Fig. Art. 7), the supplement of ACD is $(ACF + FCB) - ACD = DCB$.

THEOREM I.

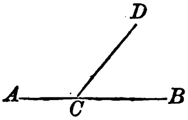
7. *The sum of all the angles formed at a point on one side of a straight line, in the same plane, is equal to two right angles.*



Let DC and EC meet the straight line AB at the point C; then $ACD + DCE + ECB =$ two right angles.

At C erect the perpendicular, CF; then it is evident that

$$ACD + DCE + ECB = ACD + DCF + FCE + ECB$$
$$= ACF + FCB = \text{two right angles.}$$



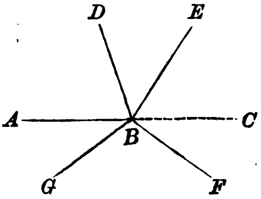
8. *Corollary* 1. If only two angles are formed, each is the supplement of the other. For by the theorem,

$ACD + DCB =$ two right angles;

therefore $ACD =$ two right angles $- DCB$,

or $DCB =$ two right angles $- ACD$.

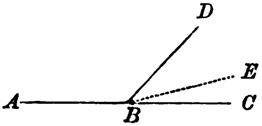
9. *Corollary* 2. The sum of all the angles formed in a plane about a point is equal to four right angles.



Let the angles ABD, DBE, EBF, FBG, GBA, be formed in the same plane about the point B. Produce AB; then the sum of the angles above the line AC is equal to two right angles; and also, the sum of the angles below the line AC is equal to two angles (7)[*]; therefore the sum of all the angles at the point B is equal to four right angles.

THEOREM II.

10. *If at a point in a straight line two other straight lines upon opposite sides of it make the sum of the adjacent angles equal to two right angles, these two lines form a straight line.*



Let the straight line DB meet the two lines, AB, BC, so as to make $ABD + DBC =$ two right angles: then AB and BC form a straight line.

For if AB and BC do not form a straight line, draw BE so that AB and BE shall form a straight line; then

$$ABD + DBE = \text{two right angles } (7);$$

but by hypothesis,

$$ABD + DBC = \text{two right angles};$$

therefore $$DBE = DBC$$

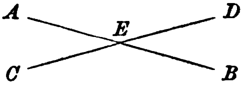
the part equal to the whole, which is absurd (Axiom 6); therefore AB and BC form a straight line.

THEOREM III.

11. *If two straight lines cut each other, the vertical angles are equal.*

Let the two lines, AB, CD, cut each other at E; then

$$AEC = DEB.$$



For AED is the supplement of both AEC and DEB (8); therefore

$$AEC = DEB$$

In the same way it may be proved that

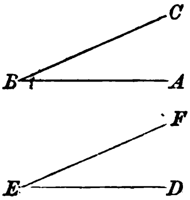
$$AED = CEB$$

---

[*] The figures alone refer to an article in the same Book; in referring to an article in another Book the number of the Book is prefixed.

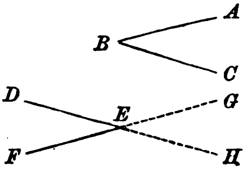
THEOREM IV.

12. *Two angles whose sides have the same or opposite directions are equal.*



1st. Let BA and BC, including the angle B, have respectively the same direction as ED and EF, including the angle E; then angle $B =$ angle E.

For since BA has the same direction as ED, and BC the same as EF, the difference of direction of BA and BC must be the same as the difference of direction of ED and EF; that is, angle $B =$ angle E.



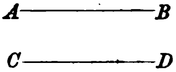
2d. Let BA and BC, including the angle B, have respectively opposite directions to ED and EF, including the angle E; then angle $B =$ angle E.

Produce DE and FE so as to form the angle GEH; then (11)

$$GEH = DEF$$

and $GEH = ABC$ by the first part of this proposition; therefore angle $B =$ angle E.

PARALLEL LINES.



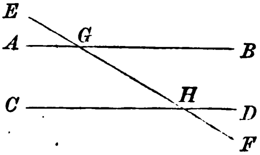
13. *Definition.* **Parallel Lines** are such as have the same direction; as AB and CD.

14. *Corollary.* Parallel lines can never meet. For, since parallel lines have the same direction, if they coincided at one point, they would coincide throughout and form one and the same straight line.

Conversely, straight lines in the same plane that never meet, however far produced, are parallel. For if they never meet they cannot be approaching in either direction, that is, they must have the same direction.

15. *Axiom.* Two lines parallel to a third are parallel to each other.

16. *Definition.* When parallel lines are cut by a third, the angles without the parallels are called *external;* those within, *internal;* thus, AGE, EGB, CHF, FHD are *external* angles; AGH, BGH, GHC, GHD are *internal* angles. Two internal angles on the same side of the secant, or cutting line, are called *internal angles on the same side;* as AGH and GHC, or BGH and GHD. Two internal angles on opposite sides of the secant, and not adjacent, are called *alternate internal angles;* as AGH and GHD, or BGH and GHC.



Two angles, one external, one internal, on the same side of the secant, and not adjacent, are called *opposite external and internal angles;* as EGA and GHC, or EGB and GHD.

THEOREM V.

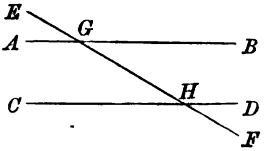
17. *If a straight line cut two parallel lines,*

1st. *The opposite external and internal angles are equal.*

2d. *The alternate internal angles are equal.*

3d. *The internal angles on the same side are supplements of each other.*

Let EF cut the two parallels AB and CD; then



1st. The opposite external and internal angles, EGA and GHC, or EGB and GHD, are equal, since their sides have the same direction (12).

2d. The alternate internal angles, AGH and GHD, or BGH and GHC, are equal, since their sides have opposite directions (12).

3d. The internal angles on the same side, *AGH* and *GHC*, or *BGH* and *GHD*, are supplements of each other; for *AGH* is the supplement of *AGE* (8), which has just been proved equal to *GHC*. In the same way it may be proved that *BGH* and *GHD* are supplements of each other.

THEOREM VI.

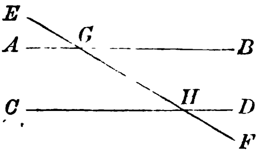
CONVERSE OF THEOREM V.

18. *If a straight line cut two other straight lines in the same plane, these two lines are parallel,*

1st. *If the opposite external and internal angles are equal.*

2d. *If the alternate internal angles are equal.*

3d. *If the internal angles on the same side are supplements of each other.*



Let *EF* cut the two lines *AB* and *CD* so as to make *EGB* = *GHD*, or *AGH* = *GHD*, or *BGH* and *GHD* supplements of each other; then *AB* is parallel to *CD*.

For, if through the point *G* a line is drawn parallel to *CD*, it will make the opposite external and internal angles equal, and the alternate internal angles equal, and the internal angles on the same side equal (17); therefore it must coincide with *AB*; that is, *AB* is parallel to *CD*.

PLANE FIGURES.

DEFINITIONS.

19. A **Plane Figure** is a portion of a plane bounded by lines either straight or curved.

When the bounding lines are straight, the figure is a *polygon*, and the sum of the bounding lines is the *perimeter*.

20. An **Equilateral Polygon** is one whose sides are equal each to each.

21. An **Equiangular Polygon** is one whose angles are equal each to each.

22. Polygons whose sides are respectively equal are *mutually equilateral.*

23. Polygons whose angles are respectively equal are *mutually equiangular.*

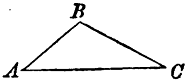
Two equal sides, or two equal angles, one in each polygon, similarly situated, are called *homologous* sides, or angles.

24. **Equal Polygons** are those which, being applied to each other, exactly coincide.

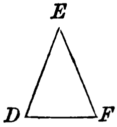
25. Of Polygons, the simplest has three sides, and is called a *triangle;* one of four sides is called a *quadrilateral;* one of five, a *pentagon;* one of six, a *hexagon;* one of eight, an *octagon;* one of ten, a *decagon.*

TRIANGLES.

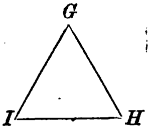
26. A **Scalene Triangle** is one which has no two of its sides equal; as *A B C.*



27. An **Isosceles Triangle** is one which has two of its sides equal; as *D E F.*

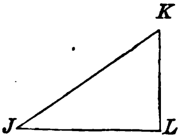


28. An **Equilateral Triangle** is one whose sides are all equal; as *I G H.*

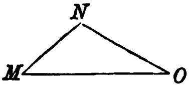


29. A **Right Triangle** is one which has a right angle; as $J K L$.

The side opposite the right angle is called the *hypothenuse*.



30. An **Obtuse-angled Triangle** is one which has an obtuse angle; as $M N O$.



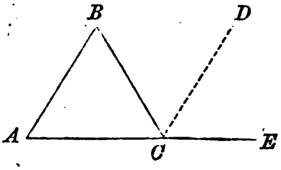
31. An **Acute-angled Triangle** is one whose angles are all acute; as $D E F$.

Acute and obtuse-angled triangles are called *oblique-angled triangles.*

32. The side upon which any polygon is supposed to stand is generally called its *base;* but in an isosceles triangle, as $D E F$, in which $D E = E F$, the third side $D F$ is always considered the base.

THEOREM VII.

33. *The sum of the angles of a triangle is equal to two right angles.*



Let $A B C$ be a triangle; the sum of its three angles, A, B, C, is equal to two right angles.

Produce $A C$, and draw $C D$ parallel to $A B$; then $D C E = A$, being external internal angles (17); $B C D = B$, being alternate internal angles (17); hence

$$D C E + B C D + B C A = A + B + B C A$$

but $D C E + B C D + B C A =$ two right angles (7);

therefore $A + B + B C A =$ two right angles.

34. *Cor.* 1. If two angles of a triangle are known, the third can be found by subtracting their sum from two right angles.

35. *Cor.* 2. If two triangles have two angles respectively equal, the remaining angles are equal.

36. *Cor.* 3. In a triangle there can be but one right angle, or one obtuse angle.

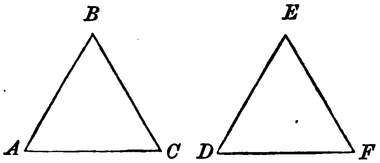
37. *Cor.* 4. In a right triangle the sum of the two acute angles is equal to a right angle.

38. *Cor.* 5. Each angle of an equiangular triangle is equal to one third of two right angles, or two thirds of one right angle.

39. *Cor.* 6. If any side of a triangle is produced, the exterior angle is equal to the sum of the two interior and opposite.

THEOREM VIII.

40. *If two triangles have two sides and the included angle of the one respectively equal to two sides and the included angle of the other, the two triangles are equal in all respects.*

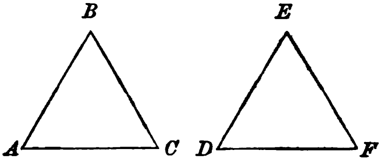


In the triangles ABC, DEF, let the side AB equal DE, AC equal DF, and the angle A equal the angle D; then the triangle ABC is equal in all respects to the triangle DEF.

Place the side AB on its equal DE, with the point A on the point D, the point B will be on the point E, as AB is equal to DE; then, as the angle A is equal to the angle D, AC will take the direction DF, and as AC is equal to DF, the point C will be on the point F; and BC will coincide with EF. Therefore the two triangles coincide, and are equal in all respects.

THEOREM IX.

41. *If two triangles have two angles and the included side of the one respectively equal to two angles and the included side of the other, the two triangles are equal in all respects.*

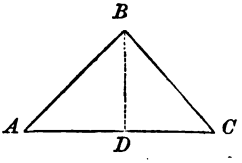


In the triangles *A B C* and *D E F*, let the angle *A* equal the angle *D*, the angle *C* equal the angle *F*, and the side *A C* equal *D F*; then the triangle *A B C* is equal in all respects to the triangle *D E F*.

Place the side *A C* on its equal *D F*, with the point *A* on the point *D*, the point *C* will be on the point *F*, as *A C* is equal to *D F*; then, as the angle *A* is equal to the angle *D*, *A B* will take the direction *D E*; and as the angle *C* is equal to the angle *F*, *C B* will take the direction *F E*; and the point *B* falling at once in each of the lines *D E* and *F E* must be at their point of intersection *E*. Therefore the two triangles coincide, and are equal in all respects.

THEOREM X.

42. *In an isosceles triangle the angles opposite the equal sides are equal.*



In the isosceles triangle *A B C* let *A B* and *B C* be the equal sides; then the angle *A* is equal to the angle *C*.

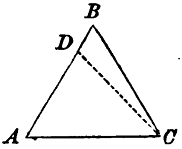
Bisect the angle *A B C* by the line *B D*; then the triangles *A B D* and *B C D* are equal, since they have the two sides *A B*, *B D*, and the included angle *A B D* equal respectively to *B C*, *B D*, and the included angle *D B C* (40); therefore the angle $A = C$.

43. *Cor.* 1. From the equality of the triangles *A B D* and *BCD*, $AD = DC$, and the angle $ADB = BDC$; that is, the line that bisects the angle opposite the base of an isosceles triangle bisects the base at right angles; also, the perpendicular bisecting the base of an isosceles triangle bisects the triangle. And, conversely, the perpendicular bisecting the base of an isosceles triangle bisects the angle opposite, and also the triangle.

44. *Cor.* 2. An equilateral triangle is equiangular.

THEOREM XI.

45. *If two angles of a triangle are equal, the sides opposite are also equal.*



In the triangle ABC, let the angle A equal the angle C; then AB is equal to BC.

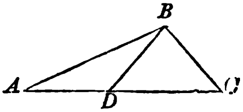
For if AB is not equal to BC, suppose AB to be greater than BC, and from AB cut off AD equal to BC and join DC. The triangles ADC and ABC have the two sides AD, AC, and the included angle A, respectively equal to the two sides BC, AC, and the included angle BCA; therefore the triangle ADC is equal to the triangle ABC (40), the part equal to the whole, which is absurd.

In the same way it can be shown that AB is not less than BC; therefore AB is equal to BC.

46. *Cor.* An equiangular triangle is equilateral.

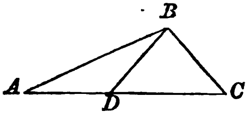
THEOREM XII.

47. *The greater side of a triangle is opposite the greater angle; and, conversely, the greater angle is opposite the greater side.*



In the triangle ABC let B be greater than C; then the side AC is greater than AB.

At the point B make the angle CBD equal to the angle C; then (45)



$$DB = DC$$

and $AC = AD + DC = AD + DB$

But (Axiom 9)

$$AD + DB > AB$$

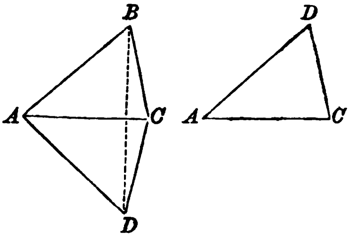
therefore $AC > AB$

Conversely. Let $AC > AB$; then the angle $ABC > C$.

For if the angle ABC is not greater than the angle C, it must be either equal to it or less. It cannot be equal, because then the side $AB = AC$ (45), which is contrary to the hypothesis. It cannot be less, because then, by the former part of this theorem, $AB < AC$, which is contrary to the hypothesis. Hence, the angle $ABC > C$.

THEOREM XIII.

48. *Two triangles mutually equilateral are equal in all respects.*



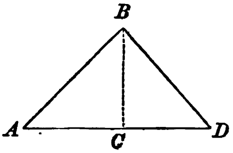
Let the triangle ABC have AB, BC, CA respectively equal to AD, DC, CA of the triangle ADC; then ABC is equal in all respects to ADC.

Place the triangle ADC so that the base AC will coincide with its equal AC, but so that the vertex D will be on the side of AC, opposite to B. Join BD. Since by hypothesis $AB = AD$, ABD is an isosceles triangle; and the angle $ABD = ADB$ (42); also, since $BC = CD$, BCD is an isosceles triangle; and the angle $DBC = CDB$; therefore the whole angle $ABC = ADC$; therefore the triangles ABC and ADC, having two sides and the included angle of the one equal to two sides and the included angle of the other, are equal (40).

49. *Scholium.* In equal triangles the equal angles are opposite the equal sides.

THEOREM XIV.

50. *Two right triangles having the hypothenuse and a side of the one respectively equal to the hypothenuse and a side of the other are equal in all respects.*



Let $A\,B\,C$ have the hypothenuse $A\,B$ and the side $B\,C$ equal to the hypothenuse $B\,D$ and the side $B\,C$ of $B\,C\,D$; then are the two triangles equal in all respects.

Place the triangle $B\,C\,D$ so that the side $B\,C$ will coincide with its equal $B\,C$, then $C\,D$ will be in the same straight line with $A\,C$ (10). An isosceles triangle $A\,B\,D$ is thus formed, and $B\,C$ being perpendicular to the base divides the triangle into the two equal triangles $A\,B\,C$ and $B\,C\,D$ (43).

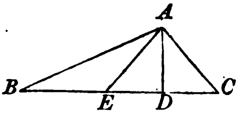
THEOREM XV.

51. *If from a point without a straight line a perpendicular and oblique lines be drawn to this line,*

1st. *The perpendicular is shorter than any oblique line.*

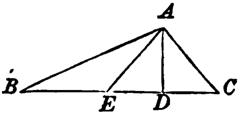
2d. *Any two oblique lines equally distant from the perpendicular are equal.*

3d. *Of two oblique lines the more remote is the greater.*



Let A be the given point, $B\,C$ the given line, $A\,D$ the perpendicular, and $A\,E$, $A\,B$, $A\,C$ oblique lines.

1st. In the triangle $A\,D\,E$, the angle $A\,D\,E$ being a right angle is greater than the angle $A\,E\,D$; therefore $A\,D < A\,E$ (47).



2d. If $DE = DC$; then the two triangles ADE and ADC, having two sides AD, DE, and the included angle ADE respectively equal to the two sides AD, DC, and the included angle ADC, are equal (40), and AE is equal to AC.

3d. If $DB > DE$; then, since AED is an acute angle, AEB is obtuse, and must therefore be greater than ABE (36); hence $AB > AE$ (47).

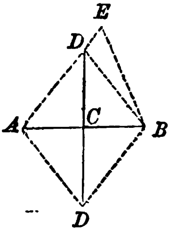
52. *Corollary.* Two equal oblique lines are equally distant from the perpendicular.

THEOREM XVI.

53. *If at the middle of a straight line a perpendicular is drawn,*

1st. *Any point in the perpendicular is equally distant from the extremities of the line.*

2d. *Any point without the perpendicular is unequally distant from the same extremities.*



Let CD be the perpendicular at the middle of the line AB; then

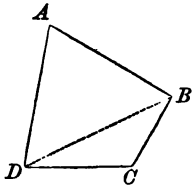
1st. Let D be any point in the perpendicular; draw DA and DB. Since $CA = CB$, $DA = DB$ (51).

2d. Let E be any point without the perpendicular; draw EA and EB, and from the point D, where EA cuts DC, draw DB. The angle $ABE > ABD = BAD$; hence, in the triangle AEB, since the angle $ABE > BAD$, $EA > EB$ (47).

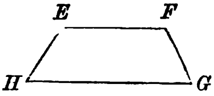
QUADRILATERALS.

DEFINITIONS.

54. A **Trapezium** is a quadilateral which has no two of its sides parallel; as *A B C D.*

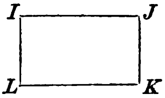


55. A **Trapezoid** is a quadrilateral which has only two of its sides parallel; as *E F G H.*

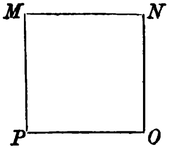


56. A **Parallelogram** is a quadrilateral whose opposite sides are parallel; as *I J K L*, or *M N O P*, or *Q R S T*, or *U V W X.*

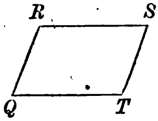
57. A **Rectangle** is a right-angled parallelogram; as *I J K L.*



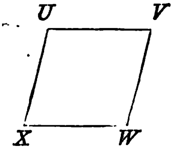
58. A **Square** is an equilateral rectangle; as *M N O P.*



59. A **Rhomboid** is an oblique-angled parallelogram; as *Q R S T.*



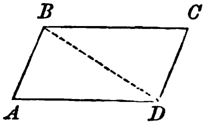
60. A **Rhombus** is an equilateral rhomboid; as *U V W X.*



61. A **Diagonal** is a line joining the vertices of two angles not adjacent; as *D B.*

THEOREM XVII.

62. *The opposite sides and angles of a parallelogram are equal to each other.*



Let $ABCD$ be a parallelogram; then will $AB = DC$, $BC = AD$, the angle $A = C$, and $B = D$.

Draw the diagonal BD. As BC and AD are parallel, the alternate angles CBD and BDA are equal (17); and as AB and DC are parallel, the alternate angles ABD and BDC are equal; therefore the two triangles ABD and BDC, having the two angles equal, and the included side BD common, are equal (41); and the sides opposite the equal angles are equal, viz.: $AB = DC$ and $BC = AD$; also the angle $A = C$, and the angle

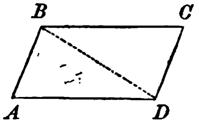
$$ABC = ABD + DBC = BDC + BDA = ADC$$

63. *Cor.* 1. The diagonal divides a parallelogram into two equal triangles.

64. *Cor.* 2. Parallels included between parallels are equal.

THEOREM XVIII.

65. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



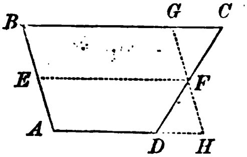
Let $ABCD$ be a quadrilateral having BC equal and parallel to AD; then $ABCD$ is a parallelogram.

Draw the diagonal BD. As BC is parallel to AD, the alternate angles CBD and BDA are equal (17); therefore the two triangles CBD and BDA, having the two sides CB, BD, and the included angle CBD respectively equal to the two sides AD, DB, and the included angle ADB, are equal (40), and DC is equal to AB, and the alternate angles ABD and BDC are equal; therefore AB is parallel to DC (18), and $ABCD$ is a parallelogram.

THEOREM XIX.

66. *The line joining the middle points of the two sides of a trapezoid which are not parallel is parallel to the two parallel sides, and equal to half their sum.*

Let EF join the middle points of the sides AB and CD, which are not parallel, of the trapezoid $ABCD$; then



1st. EF is parallel to BC and AD. Through F draw GH parallel to BA, meeting AD produced in H. The angles GFC and DFH are equal (11); also the angles GCF and FDH (17); and the side CF is equal to FD; therefore the triangles GFC and DFH are equal (41), and

$$GF = FH = \tfrac{1}{2}GH$$

But as $ABGH$ is a parallelogram, $GH = BA$ (62); therefore

$$FH = \tfrac{1}{2}BA = AE$$

therefore $AEFH$ is a parallelogram (65), and EF is parallel to AD, and therefore also to BC.

2d. $$EF = \tfrac{1}{2}(AD + BC)$$

For as $AEFH$ and $EBGF$ are parallelograms

$$EF = AH = AD + DH$$

and also $$EF = BG = BC - GC$$

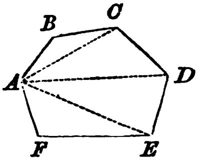
Now, as the two triangles GFC and DFH are equal, $GC = DH$; therefore, if we add the two equations, we shall have

$$2EF = AD + BC$$

or $$EF = \tfrac{1}{2}(AD + BC)$$

THEOREM XX.

67. *The sum of the interior angles of a polygon is equal to twice as many right angles as it has sides minus two.*



Let $A B C D E F$ be the given polygon; the sum of all the interior angles A, B, C, D, E, F, is equal to twice as many right angles as the figure has sides minus two.

For if from any vertex A, diagonals $A C$, $A D$, $A E$, are drawn, the polygon will be divided into as many triangles as it has sides minus two; and the sum of the angles of each triangle is equal to two right angles (33); therefore the sum of the angles of all the triangles, that is, the sum of the interior angles of the polygon, is equal to twice as many right angles as the polygon has sides minus two.

PRACTICAL QUESTIONS.

1. Do two lines that do not meet form an angle with each other? Two lines not in the same plane?

2. Does the magnitude of an angle depend upon the length of its sides?

3. If a right angle is 90°, what is the complement of an angle of 27°? of 51°? of 91°? of 153°? What is the supplement of an angle of 13°? of 83°? of 97°? of 217°?

4. If three of four angles formed at a point on the same side of a straight line, in the same plane, contain respectively 15°, 27°, and 99°, how many degrees does the fourth angle contain?

5. If five of six angles formed in a plane about a point are respectively 11°, 53°, 74°, 19°, and 117°, how many degrees are there in the sixth angle?

6. On opposite sides of a line $A B$ are two lines making with $A B$, at the point A, the first an angle of 29°, and the second an angle of 61°; how are these two lines related?

7. Can two polygons, each not equilateral, be *mutually* equilateral ?

8. Can two polygons, each not equiangular, be *mutually* equiangular ?

9. If two angles of a triangle are respectively 32° and 43°, how many degrees are there in the remaining angle ?

10. If one acute angle of a right triangle is 24°, how many degrees are there in the other acute angle ?

11. How many degrees in each angle of an equiangular triangle ?

12. How many degrees in each angle at the base of an isosceles triangle whose vertical angle is 14° ?

13. How many degrees in each acute angle of a right-angled isosceles triangle ?

14. If one of the angles at the base of an isosceles triangle is double the angle at the vertex, how many degrees in each ?

15. If the angle at the vertex of an isosceles triangle is double one of the angles at base, how many degrees in each ?

16. Two triangles mutually equilateral are equiangular (48). Are two triangles mutually equiangular also equilateral ?

17. Is a square a parallelogram ? Is a parallelogram a square ?

18. Is a rectangle a parallelogram ? Is a parallelogram a rectangle ?

19. How many sides equal to one another can there be in a trapezoid ? How many in a trapezium ?

20. How many degrees in each angle of an equiangular pentagon ? an equiangular hexagon ? octagon ? decagon ? dodecagon ?

21. If the parallel sides of a trapezoid are respectively 8 feet and 13 feet in length, how long is the line joining the middle points of the other two sides ?

22. If one of the angles of a parallelogram is 120°, how many degrees are there in each of the other angles ?

EXERCISES.

The following Theorems, depending for their demonstration upon those already demonstrated, are introduced as exercises for the pupil. In some of them references are made to the propositions upon which the demonstration depends. They are not connected with the propositions in the following books, and can be omitted if thought best.

68. Two angles whose sides have, one pair the same, the other opposite directions, are supplements of each other. (12.) (8.)

69. Any side of a triangle is less than the sum, but greater than the difference, of the other two. (*Axiom* 9.)

70. The sum of the lines drawn from a point within a triangle to the extremities of one of the sides is less than the sum of the other two sides.

Produce one of the lines to the side of the triangle. (*Axiom* 9.)

71. The angle included by the lines drawn from a point within a triangle to the extremities of one of the sides is greater than the angle included by the other two sides.

Produce as in (70). (39.)

72. The angle at the base of an isosceles triangle being one fourth of the angle at the vertex, if a perpendicular is drawn to the base at its extreme point meeting the opposite side produced, the triangle formed by the perpendicular, the side produced, and the remaining side of the triangle is equilateral.

73. If an isosceles and an equilateral triangle have the same base, and if the vertex of the inner triangle is equally distant from the vertex of the outer and the extremities of the base, then the angle at the base of the isosceles triangle is $\frac{1}{4}$ or $\frac{5}{4}$ of its vertical angle, according as it is the inner or the outer triangle.

74. Prove Theorem VII. by first drawing a line through B parallel to $A\ C$.

75. Prove Theorem VII. by drawing a triangle upon the floor, walking over its perimeter, and turning at each vertex through an angle equal to the angle at that vertex.

76. Only one perpendicular can be drawn from a point to a straight line.

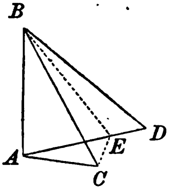
(Two cases. 1st. When the point is without the line. 2d. When the point is within the line.)

77. Two straight lines perpendicular to a third are parallel. (13.)

78. If a line joining two parallels is bisected, any other line drawn through the point of bisection and joining the parallels is bisected.

79. If two triangles have two sides of one respectively equal to two sides of the other, but the included angles unequal, the third side of the one having the included angle greater is greater than the third side of the other.

(Place the triangles as in the figure; draw *BE* bisecting the angle *CBD*, and join *C* and *E*.)



80. (Converse of 79.) If two triangles have two sides of one respectively equal to two sides of the other, but the third sides unequal, the included angle of the one having the third side greater is greater than the included angle of the other.

(Prove it by proving any other supposition absurd.)

81. Prove in Theorem XIII. the angles of the two triangles equal by reference to (79); then that the triangles are equal by (40) or (41).

82. (Converse of part of 62.) If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

83. (Converse of part of 62.) If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

84. (Converse of 63.) If a diagonal divides a quadrilateral into two equal triangles, is the figure necessarily a parallelogram?

85. The diagonals of a parallelogram bisect each other.

86. (Converse of 85.) If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

87. The diagonals of a rhombus bisect each other at right angles.

88. (Converse of 87.) If the diagonals of a quadrilateral bisect each other at right angles, the figure is a rhombus.

89. The diagonals of a rectangle are equal.

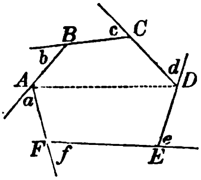
90. The diagonals of a rhombus bisect the angles of the rhombus.

91. Straight lines bisecting the adjacent angles of a parallelogram are perpendicular to each other.

92. From the vertices of a parallelogram measure equal distances upon the sides in order. The lines joining these points on the sides form a parallelogram.

93. Prove Theorem XX. by joining any point within to the vertices of the polygon.

94. If the sides of a polygon, as *ABCDEF*, are produced, the sum of the angles *a*, *b*, *c*, *d*, *e*, *f*, is equal to four right angles.



95. If a pavement is to be laid with blocks of the same regular form, prove that their upper faces must be equilateral triangles, squares, or hexagons. (67.) (9.)

96. If two kinds of regular figures, with sides of the same length, are to be used at each angular point, show that the pavement can be laid only with blocks whose upper faces are,

1st. Triangles and squares.
2d. Triangles and hexagons.
3d. Triangles and dodecagons.
4th. Squares and octagons.

How many of each must there be at each angular point?

97. If three kinds of regular figures, with sides of the same length, are to be used at each angular point, show that the pavement can be laid only with blocks whose upper faces are,

1st. Triangles, squares, and hexagons.
2d. Squares, hexagons, and dodecagons.

How many of each must there be at each angular point?

RATIO AND PROPORTION.

DEFINITIONS.

(It is necessary to understand the elementary principles of ratio and proportion before entering upon the Books that are to follow. It is therefore introduced here, but not numbered as one of the Books of Geometry, as it belongs properly to Algebra. Reference to the propositions in ratio and proportion will be made by the abbreviation Pn., with the number of the article annexed.)

1. **Ratio** is the relation of one quantity to another of the same kind; or it is the quotient which arises from dividing one quantity by another of the same kind.

Ratio is indicated by writing the two quantities after one another with two dots between, or by expressing the division in the form of a fraction. Thus, the ratio of a to b is written, $a : b$, or $\frac{a}{b}$; read, a is to b, or a divided by b.

2. The **Terms** of a ratio are the quantities compared, whether simple or compound.

The first term of a ratio is called the *antecedent*, the other the *consequent*; the two terms together are called a *couplet*.

3. An **Inverse** or **Reciprocal Ratio** of any two quantities is the ratio of their *reciprocals*. Thus, the *direct* ratio of a to b is $a : b$, that is, $\frac{a}{b}$; the *inverse* ratio of a to b is $\frac{1}{a} : \frac{1}{b}$, that is, $\frac{1}{a} \div \frac{1}{b} = \frac{b}{a}$, or $b : a$.

4. **Proportion** is an equality of ratios. Four quantities are in proportion when the ratio of the first to the second is equal to the ratio of the third to the fourth.

The equality of two ratios is indicated by the sign of equality (=), or by four dots (: :).

Thus, $a : b = c : d$, or $a : b :: c : d$, or $\frac{a}{b} = \frac{c}{d}$; read a to b equals c to d, or a is to b as c is to d, or a divided by b equals c divided by d.

5. In a proportion the antecedents and consequents of the two ratios are respectively the *antecedents* and *consequents* of the proportion. The first and fourth terms are called the *extremes*, and the second and third the *means*.

6. When three quantities are in proportion, e. g. $a : b = b : c$, the second is called a *mean proportional* between the other two; and the third, a *third proportional* to the first and second.

7. A proportion is transformed by **Alternation** when antecedent is compared with antecedent, and consequent with consequent.

8. A proportion is transformed by **Inversion** when the antecedents are made consequents, and the consequents antecedents.

9. A proportion is transformed by **Composition** when in each couplet the sum of the antecedent and consequent is compared with the antecedent or with the consequent.

10. A proportion is transformed by **Division** when in each couplet the difference of the antecedent and consequent is compared with the antecedent or with the consequent.

11. *Axiom.* Two ratios respectively equal to a third are equal to each other.

THEOREM I.

12. *In a proportion the product of the extremes is equal to the product of the means.*

Let $a : b = c : d$

that is $\frac{a}{b} = \frac{c}{d}$

Clearing of fractions $a\,d = b\,c$

13. *Scholium.* A proportion is an equation; and making the product of the extremes equal to the product of the means is merely clearing the equation of fractions.

THEOREM II.

14. *If the product of two quantities is equal to the product of two others, the factors of either product may be made the extremes, and the factors of the other the means of a proportion.*

Let $a\,d = b\,c$

Dividing by $b\,d$ $\frac{a}{b} = \frac{c}{d}$

that is $a : b = c : d$

THEOREM III.

15. *If four quantities are in proportion, they will be in proportion by alternation.*

Let $a : b = c : d$

By (12) $a\,d = b\,c$

By (14) $a : c = b : d$

THEOREM IV.

16. *If four quantities are in proportion, they will be in proportion by inversion.*

Let $a : b = c : d$

By (12) $ad = bc$

By (14) $b : a = d : c$

THEOREM V.

17. *If four quantities are in proportion, they will be in proportion by composition.*

Let $a : b = c : d$

that is $\frac{a}{b} = \frac{c}{d}$

Adding 1 to each member $\frac{a}{b} + 1 = \frac{c}{d} + 1$

or $\frac{a+b}{b} = \frac{c+d}{d}$

that is $a + b : b = c + d : d$

THEOREM VI.

18. *If four quantities are in proportion, they will be in proportion by division.*

Let $a : b = c : d$

that is $\frac{a}{b} = \frac{c}{d}$

Subtracting 1 from each member $\frac{a}{b} - 1 = \frac{c}{d} - 1$

or $\frac{a-b}{b} = \frac{c-d}{d}$

that is $a - b : b = c - d : d$

19. *Corollary.* From (17) and (18), by means of (15) and (11),

If $$a : b = c : d$$

then $$a + b : a - b = c + d : c - d$$

THEOREM VII.

20. *Equimultiples of two quantities have the same ratio as the quantities themselves.*

For $$\frac{a}{b} = \frac{ma}{mb}$$

that is $$a : b = ma : mb$$

21. *Corollary.* It follows that either couplet of a proportion may be multiplied or divided by any quantity, and the resulting quantities will be in proportion. And since by (15), if $a : b = ma : mb$, $a : m \quad a \; b : mb$ or $ma : a = mb : b$, it follows that both consequents, or both antecedents, may be multiplied or divided by any quantity, and the resulting quantities will be in proportion.

THEOREM VIII.

22. *If four quantities are in proportion, like powers or like roots of these quantities will be in proportion.*

Let $$a : b = c : d$$

that is $$\frac{a}{b} = \frac{c}{d}$$

Hence $$\frac{a^n}{b^n} = \frac{c^n}{d^n}$$

that is $$a^n : b^n = c^n : d^n$$

Since n may be either integral or fractional, the theorem is proved.

THEOREM IX.

23. *If any number of quantities are proportional, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $a : b = c : d = e : f$

Now $ab = ab$ (A)

and by (12) $ad = bc$ (B)

and also $af = be$ (C)

Adding (A), (B), (C) $a(b + d + f) = b(a + c + e)$

Hence, by (14) $a : b = a + c + e : b + d + f$

THEOREM X.

24. *If there are two sets of quantities in proportion, their products, or quotients, term by term, will be in proportion.*

Let $a : b = c : d$

and $e : f = g : h$

By (12) $ad = bc$ (A)

and $eh = fg$ (B)

Multiplying (A) by (B) $adeh = bcfg$ (C)

Dividing (A) by (B) $\frac{ad}{eh} = \frac{bc}{fg}$ (D)

From (C) by (14) $ae : bf = cg : dh$

and from (D) $\frac{a}{e} : \frac{b}{f} = \frac{c}{g} : \frac{d}{h}$

BOOK II.

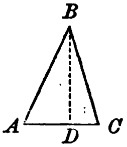
RELATIONS OF POLYGONS.

DEFINITIONS.

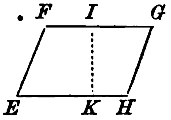
1. The **Area** of a polygon is the measure of its surface. It is expressed in units, which represent the number of times the polygon contains the square unit that is taken as a standard.

2. **Equivalent Polygons** are those which have the same area.

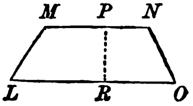
3. The **Altitude** of a triangle is the perpendicular distance from the opposite vertex to the base; as *B D*.



4. The **Altitude** of a parallelogram is the perpendicular distance from the opposite side to the base; as *I K*.



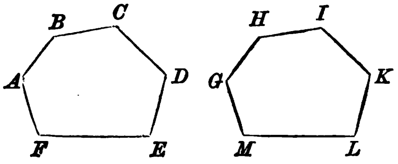
5. The **Altitude** of a trapezoid is the perpendicular distance between its parallel sides; as *P R*.

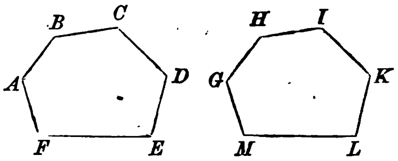


THEOREM I.

6. *Two polygons mutually equiangular and equilateral are equal.*

Let *A B C D E F* and *G H I K L M* be two polygons having the sides *A B*, *B C*, *C D*, *D E*, *E F*, *F A* and the angles *A*, *B*,



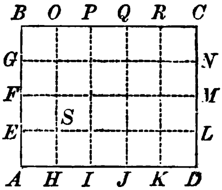


C, D, E, F of the one respectively equal to the sides $G H$, $H I$, $I K$, $K L$, $L M$, $M G$, and the angles G, H, I, K, L, M of the other; then is the polygon $A B C D E F$ equal to the polygon $G H I K L M$.

For if the polygon $A B C D E F$ is applied to the polygon $G H I K L M$ so that $A B$ shall be on $G H$ with the point A on G, B will fall on H, as $A B$ and $G H$ are equal; and as the angle B is equal to the angle H, $B C$ will take the direction $H I$; and as $B C$ is equal to $H I$, the point C will fall on I; and so also the points D, E, F will fall on the points K, L, M; and the polygon $A B C D E F$ will coincide with the polygon $G H I K L M$, and therefore be equal to it.

THEOREM II.

7. *The area of a rectangle is equal to the product of its base and altitude.*



Let $A B C D$ be a rectangle; its area $= A D \times A B$.

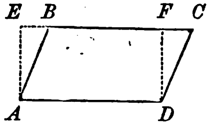
Suppose $A B$ and $A D$ to be divided into any number of equal parts, $A E$, $E F$, $A H$, $H I$, &c., and through the points of division, lines $E L$, $F M$, $H O$, $I P$, &c. be drawn parallel to the sides of the rectangle; then the rectangle will be divided into squares; these squares will be equal to each other (6). If one of the equal parts, $A E$, represents the linear unit, then one of the squares, $A E S H$, represents the square unit; and there will be as many square units in the rectangle $A E L D$ as there are linear units in $A D$; and as many square units in the rectangle $A B C D$ as there are square units in $A E L D$ multiplied by the number representing the number of linear units in $A B$; that is, the area of the rectangle is equal to the product of its base and altitude, that is $= A D \times A B$.

8. *Scholium.* If $A D$ and $A B$ have no common measure, the linear unit may be taken as small as we please, that is, so small that the remainders will be infinitesimal, and can be neglected.

9. *Corollary.* The area of a square is the square of one of its sides.

THEOREM III.

10. *The area of a parallelogram is equal to the product of its base and altitude.*



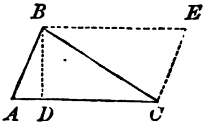
Let $D F$ be the altitude of the parallelogram $A B C D$; then the area of $A B C D = A D \times D F$.

At A draw the perpendicular $A E$ meeting $C B$ produced in E; $A E F D$ is a rectangle equivalent to the parallelogram $A B C D$. For the two triangles $A E B$ and $D F C$, having the sides $A E$, $A B$ equal respectively to the sides $D F$, $D C$ (I. 64), and the included angle $E A B$ equal to the included angle $F D C$ (I. 12), are equal. Adding $D F C$ to the common part $A B F D$ gives the parallelogram $A B C D$; and adding its equal $A E B$ to the common part $A B F D$, gives the rectangle $A E F D$; therefore the parallelogram $A B C D$ is equivalent to the rectangle $A E F D$; but the area of the rectangle $= A D \times D F$ (7); therefore the area of the parallelogram $= A D \times D F$.

THEOREM IV.

11. *The area of a triangle is equal to half the product of its base and altitude.*

Let $B D$ be the altitude of the triangle $A B C$; then the area of $A B C = \frac{1}{2} A C \times B D$.



Draw CE parallel to AB, and BE parallel to AC, forming the parallelogram $ABEC$. The triangle ABC is one half the parallelogram $ABEC$ (I. 63); the area of the parallelogram $= AC \times BD$ (10); therefore the area of the triangle $= \frac{1}{2} AC \times BD$.

12. *Cor.* 1. Triangles are to each other as the products of their bases and altitudes. For if A and a represent the altitudes of two triangles T and t, and B and b their bases, their areas are $\frac{1}{2} A \times B$ and $\frac{1}{2} a \times b$; therefore

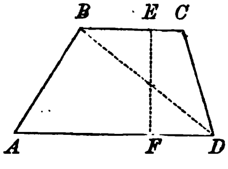
$$T : t = \tfrac{1}{2} A \times B : \tfrac{1}{2} a \times b$$

or (Pn. 21) $\quad T : t = A \times B : a \times b$

13. *Cor.* 2. Triangles having equal bases are as their altitudes; those having equal altitudes as their bases. For in the proportion above, if $B = b$, or $A = a$, the equals can be cancelled from the second ratio (Pn. 21).

THEOREM V.

14. *The area of a trapezoid is equal to half the product of its altitude and the sum of its parallel sides.*



Let EF be the altitude of the trapezoid $ABCD$; then the area of $ABCD$ $= \frac{1}{2} EF \times (BC + AD)$.

Draw the diagonal BD; it will divide the trapezoid into two triangles, ABD, BCD, having the same altitude EF as the trapezoid.

By (11) the area of $\quad BCD = \frac{1}{2} EF \times BC$

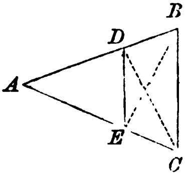
and the area of $\quad ABD = \frac{1}{2} EF \times AD$

Therefore the area of the trapezoid $= \frac{1}{2} EF \times (BC + AD)$.

15. *Corollary.* As (I. 66) the line joining the middle points of the sides AB and CD of the trapezoid $= \frac{1}{2} (BC + AD)$, therefore the area of a trapezoid is equal to the product of its altitude and the line joining the middle points of the sides which are not parallel.

THEOREM VI.

16. *A line drawn parallel to one side of a triangle divides the other sides proportionally.*



In the triangle ABC let DE be drawn parallel to BC; then

$$AE : EC = AD : DB$$

Draw DC and BE; the triangles ADE and EDC, having the same vertex D, have the same altitude; therefore (13)

$$ADE : EDC = AE : EC$$

And the triangles ADE and DEB, having the same vertex E, have the same altitude; therefore (13)

$$ADE : DEB = AD : DB$$

But the triangles EDC and DEB are equivalent (11), since they have the same base DE and the same altitude, viz., the perpendicular distance between the two parallels DE and BC. Therefore (Pn. 11) $AE : EC = AD : DB$

17. *Corollary.* As

$$AE : EC = AD : DB$$

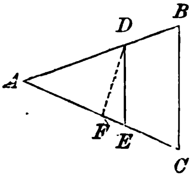
by (Pn. 17) $AE : AE + EC = AD : AD + DB$
that is $AE : AC = AD : AB$
or (Pn. 16) $AC : AE = AB : AD$

THEOREM VII.

CONVERSE OF THEOREM VI.

18. *A line dividing two sides of a triangle proportionally is parallel to the third side of the triangle.*

In the triangle ABC if DE divides AB and AC so that $AE : EC = AD : DB$, then DE is parallel to BC.



For if DE is not parallel to BC, through D draw DF parallel to BC; then (16)

$$AD : DB = AF : FC$$

But by hypothesis

$$AD : DB = AE : EC$$

Therefore (Pn. 11)

$$AF : FC = AE : EC$$

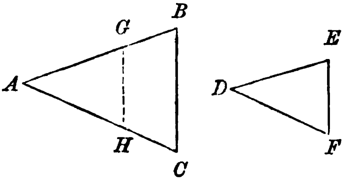
or (Pn. 15) $$AF : AE = FC : EC$$

But this proportion is absurd; for AF is less than AE, while FC is greater than EC; therefore DE is parallel to BC.

19. *Definition.* **Similar Polygons** are those which are mutually equiangular, and have their homologous sides, that is, the sides including the corresponding angles, proportional.

THEOREM VIII.

20. *Two triangles mutually equiangular are similar.*



In the two triangles ABC, DEF, let the angle $A = D$, $B = E$, and $C = F$; then the triangles are similar.

As the triangles are equiangular, we have only to prove the homologous sides proportional. Cut off AG and AH equal respectively to DE and DF, and join GH; the triangle AGH is equal to DEF (I. 40), and the angle $AGH = E$; but $E = B$; therefore $AGH = B$, and GH is parallel to BC (I. 18); and (17)

$$AB : AG = AC : AH$$

or $$AB : DE = AC : DF.$$

In like manner it may be proved that

$$AB : DE = BC : EF = AC : DF$$

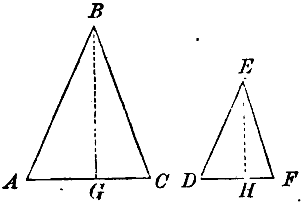
21. *Corollary.* Two triangles whose sides are equally inclined to each other are similar. For if one of the triangles is turned through an angle equal to the angle of inclination of the sides, the sides of the triangles become respectively parallel; they are therefore equiangular (I. 12) and similar (20).

THEOREM IX.

22. *The altitudes of two similar triangles are proportional to the homologous sides.*

Let BG and EH be the altitudes of the similar triangles ABC and DEF; then

$$BG : EH = AB : DE = AC : DF = BC : EF$$



For the two right triangles ABG, DEH are equiangular (I. 35), and similar (20); therefore

$$BG : EH = AB : DE = AC : DF = BC : EF$$

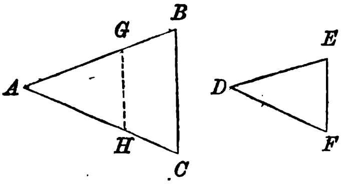
THEOREM X.

23. *Two triangles having an angle of the one equal to an angle of the other, and the sides including these angles proportional, are similar.*

In the triangles ABC, DEF let the angle $A = D$ and

$$AB : DE = AC : DF$$

then the triangles ABC and DEF are similar.



Cut off AG and AH respectively equal to DE and DF, and join GH; the triangle $AGH = DEF$, and the angle $AGH = E$ (I. 40).

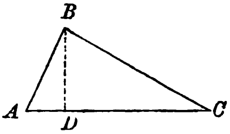
By hypothesis $\quad AB : DE = AC : DF$

or $\quad AB : AG = AC : AH$

that is, the sides AB, AC are divided proportionally by the line GH; therefore GH is parallel to BC (18), and the angle $AGH = B$ (I. 18); but the angle $AGH = E$; therefore $B = E$, and the two triangles are mutually equiangular and therefore similar (20).

THEOREM XI.

24. *In a right triangle the perpendicular drawn from the vertex of the right angle to the hypothenuse divides the triangle into two triangles similar to the whole triangle and to each other.*



In the right triangle ABC if BD is drawn from the vertex B of the right angle to the hypothenuse AC, the two triangles ABD, BCD are similar to ABC and to each other.

The two right triangles ABD and ABC have the acute angle A common; they are therefore equiangular (I. 35), and similar (20). The two right triangles ABC and BCD have the acute angle C common; therefore they are equiangular and similar. The two triangles ABD and BCD, being each similar to ABC, are similar to each other.

25. *Cor.* 1. Since ABC and ABD are similar triangles

$$AC : AB = AB : AD$$

And since ABC and BCD are similar

$$AC : CB = CB : CD$$

that is, *in a right triangle either side about the right angle is a mean proportional between the whole hypothenuse and the segment adjacent to that side cut off by the perpendicular drawn from the vertex of the right angle to the hypothenuse.*

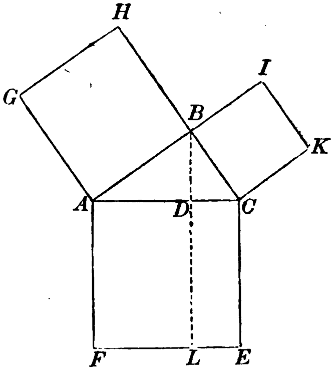
26. *Cor.* 2. As ABD and BCD are similar triangles

$$AD : DB = DB : DC$$

that is, *in a right triangle the perpendicular from the vertex of the right angle is a mean proportional between the segments of the base.*

THEOREM XII.

27. *The square described on the hypothenuse of a right angle is equivalent to the sum of the squares described upon the other two sides.*



Let ABC be a triangle right-angled at B; then

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

On the three sides construct squares, draw BD perpendicular to AC, and produce it to FE; $DCEL$ is a rectangle whose area is (7)

$$CE \times CD = AC \times CD$$

The area of the square (9)

$$BIKC = \overline{BC}^2$$

But (25) $$AC : BC = BC : CD$$

or $$AC \times CD = \overline{BC}^2$$

that is, the square $BIKC$ is equivalent to the rectangle $DCEL$. In the same way the square $AGHB$ can be proved equivalent to the rectangle $ADLF$: therefore the sum of the two rectangles, that is, the square $ACEF$ is equivalent to the sum of the squares $BIKC$ and $AGHB$; or

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

28. *Corollary.* Since

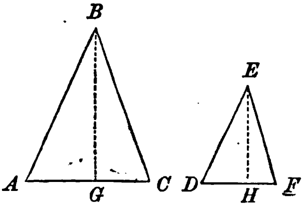
$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

$$\overline{BC}^2 = \overline{AC}^2 - \overline{AB}^2$$

and $$BC = \sqrt{\overline{AC}^2 - \overline{AB}^2}$$

THEOREM XIII.

29. *Similar triangles are to each other as the squares of their homologous sides.*



Let ABC and DEF be two similar triangles; then

$$ABC : DEF = \overline{AC}^2 : \overline{DF}^2$$

Draw BG and EH perpendicular respectively to AC and DF; then (22)

$$BG : EH = AC : DF$$

this multiplied by the proportion

$$\tfrac{1}{2}\,AC : \tfrac{1}{2}\,DF = AC : DF$$

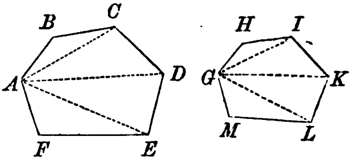
gives $\quad \tfrac{1}{2}\,AC \times BG : \tfrac{1}{2}\,DF \times EH = \overline{AC}^2 : \overline{DF}^2$

but $\tfrac{1}{2}\,AC \times BG$ is the area of ABC, and $\tfrac{1}{2}\,DF \times EH$ is the area of DEF (11); therefore

$$ABC : DEF = \overline{AC}^2 : \overline{DF}^2$$

THEOREM XIV.

30. *Similar polygons can be divided into the same number of similar triangles.*



Let $ABCDEF$ and $GHIKLM$ be similar polygons; they can be divided into the same number of similar triangles.

From the homologous angles A and G draw the diagonals AC, AD, AE, GI, GK, and GL; these diagonals divide the polygons as required. For, as the polygons are similar, the angle $B = H$, and $AB : GH = BC : HI$; therefore the triangles ABC and GHI are similar (23). As the triangles ABC and GHI are similar, the angle $BCA = HIG$; but the whole angle $BCD = HIK$; therefore the angle $ACD = GIK$; and as the triangles ABC and GHI are similar

$$BC : HI = AC : GI$$

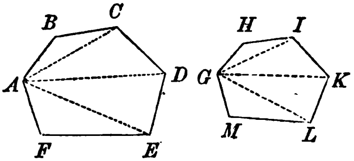
But $BC : HI = CD : IK$

Therefore $AC : GI = CD : IK$

and ACD and GIK are similar (23). In like manner it can be proved that the other triangles are similar each to each.

THEOREM XV.

31. *The perimeters of similar polygons are to each other as the homologous sides; and the polygons as the squares of the homologous sides.*



Let $ABCDEF$ and $GHIKLM$ be two similar polygons.

1st. Their perimeters are to each other as $AB : GH$

For as the polygons are similar

$$AB : GH = BC : HI = CD : IK, \text{ \&c.}$$

Therefore (Pn. 23)

$$AB + BC + CD, \text{ \&c.} : GH + HI + IK, \text{ \&c.} = AB : GH$$

that is, the perimeters of $ABCDEF$ and $GHIKLM$ are as $AB : GH$.

2d. $ABCDEF : GHIKLM = \overline{AB}^2 : \overline{GH}^2$

From the homologous angles A and G draw the diagonals AC, AD, AE, GI, GK, and GL; the polygons will be divided into the same number of similar triangles (30); therefore (29)

$$ABC : GHI = \overline{AC}^2 : \overline{GI}^2$$

and $ACD : GIK = \overline{AC}^2 : \overline{GI}^2$

Therefore $ABC : GHI = ACD : GIK$

In like manner $ACD : GIK = ADE : GKL$

and $ADE : GKL = AEF : GLM$

Hence (Pn. 23)

$$ABC + ACD + ADE + AEF : GHI + GIK + GKL + GLM = ABC : GHI$$

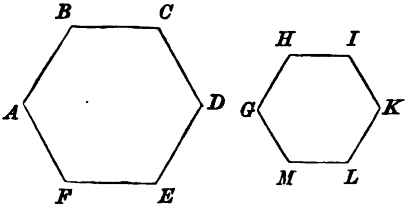
But $ABC : GHI = \overline{AB}^2 : \overline{GH}^2$

Therefore the sums of the triangles, that is, the polygons themselves, are to each other as the squares of the homologous sides.

32. *Definition.* A **Regular Polygon** is one that is both equiangular and equilateral.

THEOREM XVI.

33. *Regular polygons of the same number of sides are similar.*



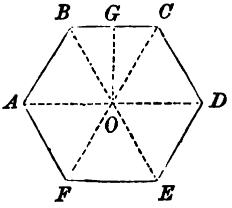
Let $ABCDEF$ and $GHIKLM$ be two regular polygons of the same number of sides; they are similar.

They are equiangular; for the sum of their angles is the same (I. 67); and each angle is equal to this sum divided by the number of angles which is the same.

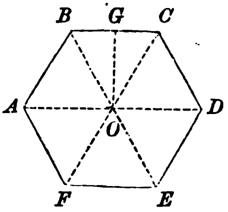
The homologous sides are proportional; for as the polygons are regular, $AB = BC = CD$, &c., and $GH = HI = IK$, &c., therefore $AB : GH = BC : HI = CD : IK$, &c.

THEOREM XVII.

34. *There is a point in a regular polygon equidistant from its vertices, and also equidistant from its sides.*



Let $ABCDEF$ be a regular polygon. Bisect the angles A and B by AO and BO. As the whole angles A and B are each less than the two right angles, the sum of OAB and ABO is less than two right angles; therefore AO and BO cannot be parallel (I. 17), but will meet.



Suppose them to meet in the point O; then O is equidistant from the vertices A, B, C, D, E, F, and also from the sides AB, BC, CD, &c.

Draw OC, OD, OE, OF. $OA = OB$ (I. 45). As OB bisects the whole angle B, the angle $OBA = OBC$; therefore the triangle $ABO = OBC$ (I. 40), and $OC = OA = OB$. In like manner it can be proved that $OD = OE = OF = OA$; that is, O is equidistant from the vertices of the polygon.

As the triangles OAB, OBC, OCD, &c. are equal, their altitudes are equal, that is, the bases are equidistant from the vertex O.

35. *Scholium.* O is called *the centre,* and the perpendicular OG *the apothem of the polygon.*

36. *Corollary.* In regular polygons of the same number of sides, the apothems are as the homologous sides; therefore *the perimeters of regular polygons of the same number of sides are as their apothems; and the polygons as the squares of their apothems.*

THEOREM XVIII.

37. *The area of a regular polygon is equal to half the product of its perimeter and apothem.*

For the area of each triangle of which the polygon is composed is equal to half the product of its base and the apothem of the polygon (11); therefore the area of the polygon is equal to half the product of the sum of the bases, that is, its perimeter and its apothem.

PRACTICAL QUESTIONS.

1. What is the perimeter and the area of a rectangle 25 by 35 inches?

2. What is the area of a parallelogram whose base is 20 feet and altitude 12 feet?

3. What is the area of a triangle whose base is 14 feet and altitude 8 feet?

4. What is the square surface of a board 15 feet long, and 16 inches wide at one end and 9 inches at the other? What kind of a figure is it?

5. What integral numbers will express the sides and hypothenuse of a right triangle?

6. How far from a tower 40 feet high must the foot of a ladder 50 feet long be placed that it may exactly reach the top of the tower?

7. The foot of a ladder 67 feet long stands 40 feet from a wall; how much nearer the wall must the foot be placed that the ladder may reach 10 feet higher?

8. If a ladder 108 feet long, with its foot in the street, will reach on one side to a window 75 feet high, and on the other to a window 45 feet high; how wide is the street?

9. A has an acre of land one of whose sides is 20 rods in length; B has a piece of land of exactly similar form containing 9 acres. What is the length of the corresponding side of B's?

10. What is the distance on the floor from one corner to the opposite corner of a rectangular room 16 by 24 feet?

11. If the height of the above room is 10 feet, what is the distance from the lower corner to the opposite upper corner?

12. Find the length of the longest straight line that can be drawn in a cube whose dimensions are 12, 4, and 3.

13. What is the altitude of an equilateral triangle whose side is 12 feet?

14. If the bases of two similar triangles are respectively 100 and 10 feet, how many triangles equal to the second are equivalent to the first?

15. How many times as much paint will it take to cover a church whose steeple is 120 feet in height as to cover an exact model of the church whose steeple is 10 feet in height?

16. What is the area of a right-angled triangle whose hypothenuse is 125 feet and one of the sides 75 feet?

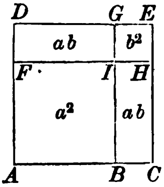
EXERCISES.

The following Theorems, depending for their demonstration upon those already demonstrated, are introduced as exercises for the pupil. In some of them references are made to the propositions upon which the demonstration depends. They are not connected with the propositions in the following books, and can be omitted if thought best.

38. The square on the sum AC of two straight lines AB, BC is equivalent to the squares on AB and BC, together with twice the rectangle $AB.BC$.

Or, algebraically, if $a = AB$, and $b = BC$,

$$(a + b)^2 = a^2 + 2\,ab + b^2$$

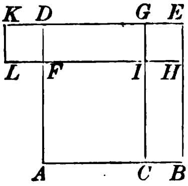


39. *Corollary.* The square on a line is four times the square on half of the line.

40. The square on the difference AC of two straight lines AB, BC is equivalent to the squares on AB and BC, diminished by twice the rectangle $AB.BC$.

Or, algebraically, if $a = AB$, and $b = BC$,

$$(a - b)^2 = a^2 - 2\,ab + b^2$$

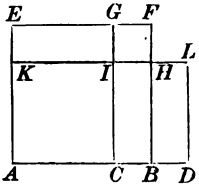


41. The rectangle contained by the sum and difference of two lines AB, BC is equivalent to the difference of their squares.

Or, algebraically, if $a = AB$ and $b = BC$

$$(a + b)\,(a - b) = a^2 - b^2$$

Produce AB so that $BD = BC$.



42. Parallelograms are to each other as the products of their bases and altitudes. (10.)

43. Parallelograms having equal bases are to each other as their altitudes; those having equal altitudes are as their bases.

44. Where must a line from the vertex be drawn to bisect a triangle? (13.)

45. Two or more lines parallel to the base of a triangle divide the other sides, or the other sides produced, proportionally.

46. Lines joining the middle points of the adjacent sides of a quadrilateral form a parallelogram; and the perimeter of this parallelogram is equal to the sum of the diagonals of the quadrilateral.

Draw the diagonals. (18.)

47. Lines drawn from the vertex of a triangle divide the opposite side and a parallel to it proportionally.

? **48.** State and prove the converse of 47.

49. $ABCD$ is a parallelogram; E and F the middle points of AB and CD. BF and DE trisect the diagonal AC.

50. If two triangles have two sides of the one equal respectively to two sides of the other, and the included angles supplementary, the triangles are equivalent.

51. The diagonals divide a parallelogram into four equivalent triangles. Two triangles standing on opposite sides are equal.

52. If the middle points of the sides of a triangle are joined, the area of the triangle thus formed is one fourth the area of the original triangle.

53. Every line passing through the intersection of the diagonals of a parallelogram bisects the parallelogram.

54. If a point within a parallelogram is joined to the vertices, the two triangles formed by the joining lines and two opposite sides are together equivalent to half the parallelogram.

Through the point draw lines parallel to the sides of the parallelogram.

55. State and prove the proposition if the point named in 54 is without the parallelogram.

56. The area of a trapezoid is equal to twice the area of the triangle formed by joining the extremities of one non-parallel side to the middle point of the other.

57. Two triangles are similar if two angles of the one are equal respectively to two angles of the other.

58. Two triangles are similar if their homologous sides are proportional.

59. *Definition.* When a point is taken on a given line, or a given line produced, the distances of the point from the extremities of the line are called the *segments.* If the point is within the given line, the sum of the segments, if in the line produced, the difference of the segments, is equal to the line.

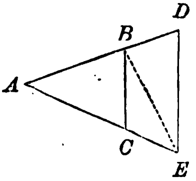
60. The line bisecting any angle, interior or exterior, divides the opposite side into segments which are proportional to the adjacent sides.

Let B be the bisected angle of a triangle ABC. Through C draw a line parallel to the bisecting line and meeting AB. If the interior angle at B is bisected, AB must be produced; if the exterior angle, AC. In the latter case, if E is the point where the bisecting line meets AC produced, the segments of the base (59) are AE and CE. (I. 17.) (I. 45.) (16.)

61. Two triangles having an angle of the one equal to an angle in the other are to each other as the rectangles of the sides containing the equal angles; or

$ABC : ADE = AB \times AC : AD \times AE$

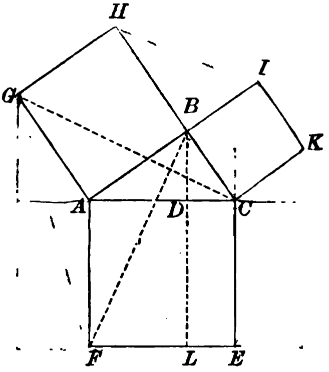
Draw BE. (13.) (Pn. 24.) (Pn. 21.)



62. Prove Theorem XII., first drawing GC and BF; then proving the triangles AGC and ABF equal.

Turn the triangle ABF on the point A in its own plane till AB coincides with AG; where will F be? (7, 11.)

63. Prove that if GH, KI, and LB, in the figure above, are produced, they will meet in the same point.

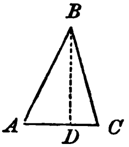


64. Prove Theorem XII., first producing FA to GH, and producing GH, KI, and LB till they meet.

65. Prove Theorem XII., first constructing the squares on opposite sides of AB and BC from that on which they are drawn in the figure in Art. 62; moving the square $AGHB$ on AB, a distance equal to BC in the direction BA; then proving that these squares are divided into parts that can be made to coincide with the parts of the square on AC.

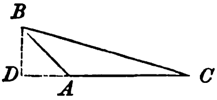
66. If A is an acute angle of the triangle ABC, and BD is the perpendicular from B to AC, then

$$BC^2 = AB^2 + AC^2 - 2\,AC \times AD$$



67. If A is an obtuse angle of the triangle ABC, and BD is the perpendicular from B to AC, then

$$BC^2 = AB^2 + AC^2 + 2\,AC \times AD$$



68. Show that if the angle A becomes a right angle, both 66 and 67 reduce to the same as 27; and if C becomes a right angle, both reduce to the same as the second equation in 28.

69. If a line is drawn from the vertex of any angle of a triangle to the middle of the opposite side, the sum of the squares of the other two sides is equivalent to twice the square of the bisecting line together with twice the square of a segment of the bisected side.

Draw a perpendicular from the same vertex to the opposite side. (66, 67.)

70. The sum of the squares of the four sides of a parallelogram is equivalent to the sum of the squares of the diagonals. (69.) (39.)

71. In the figure in Art. 62 draw HI, KE, FG. The triangle HIB is equal, and the triangles CKE, GAF are equivalent to ABC.

72. The squares of the sides of a right triangle are as the segments of the hypothenuse made by a perpendicular from the vertex of the right angle.

73. The square of the hypothenuse is to the square of either side as the hypothenuse is to the segment adjacent to this side made by a perpendicular from the vertex of the right angle.

74. The side of a square is to its diagonal as $1 : \sqrt{2}$; or the square described on the diagonal of a square is double the square itself.

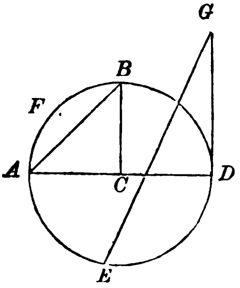
75. (Converse of 30.) Two polygons composed of the same number of similar triangles similarly situated are similar.

BOOK III.

THE CIRCLE.

DEFINITIONS.

1. A **Circle** is a plane figure bounded by a curved line called the *circumference*, every point of which is equally distant from a point within called the *centre;* as *A B D E.*



2. The **Radius** of a circle is a line drawn from the centre to the circumference; as *C D.*

3. The **Diameter** of a circle is a line drawn through the centre and terminating at both ends in the circumference; as *A D.*

4. *Corollary.* The radii of a circle, or of equal circles, are equal; also the diameters are equal, and each is equal to double the radius.

5. An **Arc** is any part of the circumference; as *A F B.*

6. A **Chord** is the straight line joining the ends of an arc; as *A B.*

7. A **Segment** of a circle is the part of the circle cut off by a chord; as the space included by the arc *A F B* and the chord *A B.*

8. A **Sector** is the part of a circle included by two radii and the intercepted arc; as the space *B C D.*

9. A **Tangent** (in geometry) is a line which touches, but does not, though produced, cut the circumference; as *G D.*

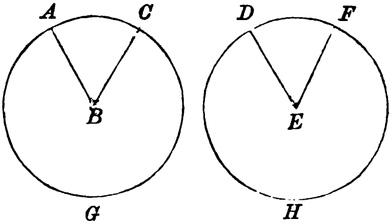
A tangent is often considered as terminating at one end at the point of contact, at the other where it meets another tangent or a secant.

10. A **Secant** (in geometry) is a line lying partly within and partly without a circle; as GE.

A secant is generally considered as terminating at one end where it meets the concave circumference, and at the other where it meets another secant or a tangent.

THEOREM I.

11. *In the same circle, or equal circles, equal angles at the centre are subtended by equal arcs; and, conversely, equal arcs subtend equal angles at the centre.*



Let B and E be equal angles at the centres of the two equal circles ACG and DFH; then the arcs AC and DF are equal.

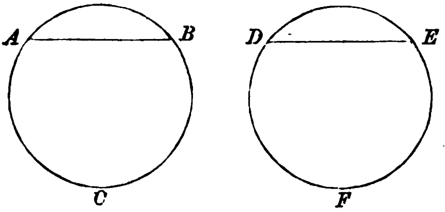
Place the angle B on the angle E; as they are equal they will coincide; and as BA and BC are equal to ED and EF, the point A will coincide with D, and the point C with F; and the arc AC will coincide with DF, otherwise there would be points in the one or the other arc unequally distant from the centre.

Conversely. If the arcs AC and DF are equal, the angles B and E are equal.

For, if the radius AB is placed on the radius DE with the point B on E, the point A will fall on D, as $AB = DE$; and the arc AC will coincide with DF, otherwise there would be points in the one or the other arc unequally distant from the centre; and as the arc $AC = DF$, the point C will fall on F; therefore BC will coincide with EF, and the angle B be equal to E.

THEOREM II.

12. *In the same or equal circles, equal chords subtend equal arcs; and, conversely, equal arcs are subtended by equal chords.*

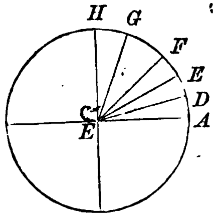


Let $A\,B\,C$ and $D\,E\,F$ be two equal circles; if the arcs $A\,B$ and $D\,E$ are equal, the chords $A\,B$ and $D\,E$ are equal; and conversely, if the chords $A\,B$ and $D\,E$ are equal, the arcs $A\,B$ and $D\,E$ are equal.

For, if the centre of the circle $A\,B\,C$ is placed on the centre of $D\,E\,F$ with the point A of the circumference on the point D, if the arcs or the chords are equal, B will fall on E; and in either case the chords and arcs will coincide, otherwise there would be points in the one or the other circumference unequally distant from the centre.

THEOREM III.

13. *Angles at the centre vary as their corresponding arcs.*



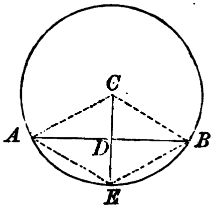
Let $A\,C\,D$, $D\,C\,E$, $E\,C\,F$ be equal angles at the centre C; then the arcs $A\,D$, $D\,E$, $E\,F$ are equal (11); then the angle $A\,C\,E$, being double the angle $A\,C\,D$, the arc $A\,E$ is double the arc $A\,D$; and the angle $A\,C\,F$, being three times the angle $A\,C\,D$, the arc $A\,F$ is three times the arc $A\,D$; and the angle $A\,C\,G$, being m times the angle $A\,C\,D$, the arc $A\,G$ is m times the arc $A\,D$; that is, the angle varies as the arc, or the arc as the angle.

14. *Cor.* 1. As angles at the centre vary as their arcs, or arcs as their corresponding angles, either of these quantities may be assumed as the measure of the other. The measure of an angle is, then, *the arc included between its sides and described from its vertex as a centre.*

15. *Cor.* 2. As the sum of all the angles about the point C is equal to four right angles (I. 9), one right angle, HCA, is measured by one quarter of the circumference, or by a quadrant.

THEOREM IV.

16. *The radius perpendicular to a chord bisects the chord and the arc subtended by the chord.*



Let CE be a radius perpendicular to the chord AB; it bisects the chord AB, and also the arc AEB.

Draw the radii CA and CB and the chords AE and EB. As equal oblique lines are equally distant from the perpendicular, $AD = DB$ (I. 52); and as E is a point in the perpendicular to the middle of AB, it is equally distant from A and B (I. 53); therefore the chords and the arcs AE, EB are equal.

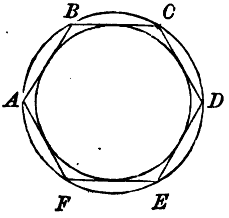
17. *Corollary.* The perpendicular to the middle of a chord passes through the centre of the circle.

DEFINITIONS.

18. An **Inscribed Angle** is one whose vertex is in the circumference and whose sides are chords; as DAB.

19. An **Inscribed Polygon** is one whose sides are chords.

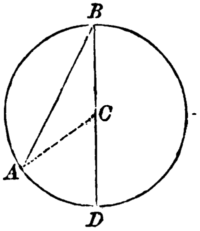
Thus *ABCDEF* is inscribed in the outer circle. In this case the circle is said to be circumscribed about the polygon.



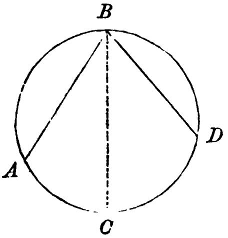
20. A **Circumscribed Polygon** is one whose sides are tangents. Thus *A B C D E F* is circumscribed about the inner circle. In th[illegible]case the circle is said to be inscribed in the polygon.

THEOREM V.

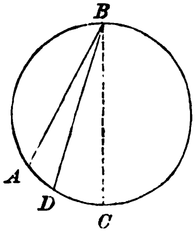
21. *An inscribed angle is measured by half the arc included by its sides.*



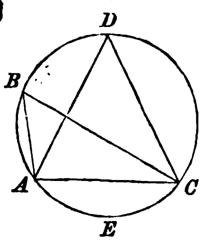
1st. When one of the sides *B D* is a diameter; then the angle *B* is measured by half the arc *A D.* Draw the radius *C A*, and the triangle *A C B* is isosceles, *C A* and *C B* being radii; therefore the angle *A* = *B* (I. 42). But the exterior angle *A C D* is equal to the sum of the two angles *A* and *B* (I. 39); therefore the angle ~~*A C D*~~ B is equal to half the angle ~~*B*~~ ACD; the angle *A C D* is measured by the arc *A D* (14); therefore the angle *B* is measured by half the arc *A D.*



2d. When the centre is within the angle, draw the diameter *B C.* By the preceding part of the proposition the angle *A B C* is measured by half the arc *A C*, and *C B D* by half *C D*; therefore *A B C* + *C B D*, or *A B D*, is measured by half *A C* + *C D*, or the arc *A D.*



3d. When the centre is without the angle, draw the diameter BC. By the first part of the proposition the angle ABC is measured by half the arc AC, and DBC by half DC; therefore $ABC - DBC$, or ABD, is measured by $AC - DC$, or the arc AD.

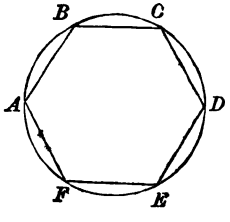


22. *Cor.* 1. All the angles ABC, ADC, inscribed in the same segment are equal; for each is measured by half the arc AEC.

23. *Cor.* 2. Every angle inscribed in a semicircle is a right angle; for it is measured by half a semi-circumference, or by a quadrant (15).

THEOREM VI.

24. *Every equilateral polygon inscribed in a circle is regular.*



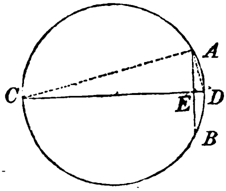
Let $ABCDEF$ be an equilateral polygon inscribed in a circle; it is also equiangular and therefore regular.

For the chords AB, BC, CD, &c. being equal, the arcs AB, BC, CD, &c. are equal (12); therefore the arc $AB +$ the arc BC will be equal to the arc $BC +$ the arc CD, &c.; that is, the angles B, C, &c. are in equal segments; therefore they are equal (22), and the polygon is equiangular and regular.

THEOREM VII.

25. *An infinitely small chord coincides with its arc.*

Let AB be an infinitely small chord; it coincides with the arc ADB.



Draw the diameter CD perpendicular to the chord AB; and draw AC and AD; CAD is a right-angled triangle (23); therefore (II. 26)

$$CE : AE = AE : ED$$

that is, ED is the same part of AE that AE is of CE. But AE is half the infinitely small chord AB (16), and AB is infinitely small in comparison with CE; therefore ED is infinitely small in comparison with AE, that is, the point E is on D, and the chord AB coincides with the arc ADB.

THEOREM VIII.

26. *A circle is a regular polygon of an infinite number of sides.*

If the circumference of a circle is divided into equal arcs, each infinitely small, the infinitely small chords of these arcs would form a regular polygon (24) of an infinite number of sides; and as each chord would coincide with its arc (25), the polygon would be the circle itself.

27. *Scholium.* It might be supposed that although the difference between each chord and its arc is infinitesimal, yet as there is an infinite number of these differences their sum would not be infinitesimal and ought not to be neglected; that is, that the perimeter of the polygon and the circumference of the circle differed by a finite quantity. But each chord is infinitely small compared with the diameter of the circle, or is equal to $\frac{D}{Inf.}$; and the difference between each chord and its arc is infinitely smaller than the chord itself, or is equal to $\frac{D}{Inf. \times Inf.}$; and an infinite number of these differences is equal to $\frac{D}{Inf. \times Inf.} \times Inf. = \frac{D}{Inf.}$; that is, the difference between the perimeter of the polygon and the circumference of the circle is infinitesimal.

THEOREM IX.

28. *Circumferences of circles are to each other as their radii, or as their diameters.*

For circles are regular polygons of an infinite number of sides (26); and if the circumferences of circles are divided into the same infinite number of arcs, the polygons formed by their chords, that is, the circles themselves, are regular polygons of the same number of sides and are therefore similar (II. 33); and the apothems of the polygons are the radii of the circles; therefore the circumferences of the circles are as their radii (II. 36), or as twice their radii, that is, as their diameters.

29. *Cor.* 1. If C and c denote the circumferences, R and r the corresponding radii, and D and d the corresponding diameters, we have

$$C : c = R : r = D : d$$

or

$$C : R = c : r$$

and

$$C : D = c : d$$

That is, the ratio of the circumference of every circle to its radius or to its diameter is the same, that is, is constant. The constant ratio of the circumference to its diameter is denoted by π (the Greek letter p).

30. *Cor.* 2.

$$\frac{C}{D} = \pi$$

$$C = \pi D = 2 \pi R$$

THEOREM X.

31. *The area of a circle is equal to half the product of its circumference and its radius.*

The area of a regular polygon is half the product of its perimeter and its apothem (II. 37); a circle is a regular polygon of an infinite number of sides (26); the circumference of the circle is the perimeter of the polygon, and its radius is the apothem; therefore the area of a circle is half the product of its circumference and its radius.

32. *Corollary.* If $C =$ the circumference, $D =$ the diameter, $R =$ the radius, and $A =$ the area of a circle, we have

$$A = \tfrac{1}{2}\, C \times R$$

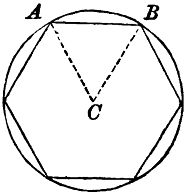
But (30) $$C = 2\,\pi\, R = \pi\, D$$

Therefore $$A = \tfrac{1}{2} \times 2\,\pi\, R \times R = \pi\, R^2$$

or $$A = \tfrac{1}{2}\,\pi\, D \times \frac{D}{2} = \tfrac{1}{4}\,\pi\, D^2$$

THEOREM XI.

33. *The side of a regular hexagon inscribed in a circle is equal to the radius of the circle.*



In the circle whose centre is C draw the chord $A\,B$ equal to the radius; $A\,B$ is the side of a regular hexagon inscribed in a circle.

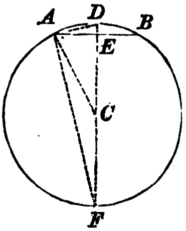
Draw the radii $C\,A$ and $C\,B$; $C\,A\,B$ is an equilateral, and therefore an equiangular triangle; hence the angle C is equal to one third of two right angles, or one sixth of four right angles; that is, the arc $A\,B$ is one sixth of the whole circumference, or the chord $A\,B$ the side of a regular hexagon inscribed in the circle (12 and 24).

34. *Corollary.* The chord of half the arc $A\,B$ would be the side of a regular dodecagon; the chord of one quarter of the arc $A\,B$, the side of a regular polygon of twenty-four sides; and so on.

PROPOSITION XII.

PROBLEM.

35. *The chord of an arc given to find the chord of half the arc.*



Let AB be the given chord, AD the chord of half the arc ADB, and R denote the radius.

Draw the diameter DF, the radius AC, and the chord AF. The triangle ADF is right angled at A (23); then (II. 25)

$$DF : AD = AD : DE$$

or $$AD^2 = DF \times DE = 2R \times DE$$

and $$AD = \sqrt{2R \times DE}$$

Now $$DE = DC - CE = R - CE$$

and (II. 28) $$CE = \sqrt{AC^2 - AE^2} = \sqrt{R^2 - AE^2}$$

therefore $$DE = R - \sqrt{R^2 - AE^2}$$

Substituting this value of DE in

$$AD = \sqrt{2R \times DE}$$

we have

$$AD = \sqrt{2R^2 - 2R\sqrt{R^2 - AE^2}}$$

36. *Cor.* 1. If C denote the given chord, c the chord of half the arc, the equation becomes

$$c = \sqrt{2R^2 - 2R\sqrt{R^2 - \frac{C^2}{4}}}$$

$$= \sqrt{2R^2 - R\sqrt{4R^2 - C^2}}$$

37. *Cor.* 2. If the diameter D, that is, $2R$, is unity, the equation in (36) becomes

$$c = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - C^2}}$$

PROPOSITION XIII.

PROBLEM.

38. *To find the arithmetical value of the constant π.*

From (30) $C = \pi D$; if $D = 1$, this equation becomes $C = \pi$. If then we can find the circumference of a circle whose diameter is unity, we shall have the value of π.

If the diameter is unity, radius is one half, and the side of a regular hexagon inscribed in the circle is one half (33), and the perimeter of the hexagon is $6 \times \frac{1}{2} = 3$.

As the diameter is unity, and the side of the inscribed hexagon one half, we can find the side of the regular inscribed dodecagon from the equation in (37):

$$
\begin{aligned}
c &= \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \tfrac{1}{4}}} \\
&= \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{.75}} \\
&= \sqrt{.5 - .433} \\
&= \sqrt{.067} = .2588+
\end{aligned}
$$

The perimeter of the inscribed dodecagon is therefore $12 \times .2588+ = 3.105+$.

By using the side of the dodecagon $= .2588+$, as C, or $.067 = C^2$, from the same equation we can find the side of a regular inscribed polygon of twenty-four sides:

$$
\begin{aligned}
c &= \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - .067}} \\
&= \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{.933}} \\
&= \sqrt{.5 - .483} \\
&= \sqrt{.017} = .13038
\end{aligned}
$$

The perimeter of the inscribed polygon of twenty-four sides is therefore $24 \times .13038 = 3.129$.

By continuing this process we approximate to the circumference, that is, to the value of π.

39. *Scholium.* By other more expeditious methods the value of π has been found accurately to two hundred and fifty places of decimals. For practical purposes it is sufficiently accurate to call $\pi = 3.14159$.

PRACTICAL QUESTIONS.

1. What is the circumference of a circle whose radius is 10 feet?

2. What is the diameter of a circle whose circumference is 57 rods?

3. What is the area of a circle whose radius is 40 feet?

4. What is the area of a circle whose circumference is 18 inches?

5. What is the circumference of a circle whose area is 116 square feet?

6. The radii of two concentric circles are 40 and 54 feet; what is the area of the space bounded by their circumferences?

7. A has a circular lot of land whose diameter is 95 rods, and B a similar lot whose area is 750 square rods; compare these lots.

8. What is the difference between the perimeters of two lots of land each containing an acre, if one is a square and the other a circle?

9. What is the area of a square inscribed in a circle whose area is a square metre?

10. What is the area of a regular hexagon inscribed in a circle whose area is 567 square feet.

11. If a rope an inch in diameter will support 1,000 pounds, what must be the diameter of a rope of like material to support 4,000 pounds?

12. If a pipe an inch in diameter will fill a cistern in 25 minutes, how long will it take a pipe 5 inches in diameter?

13. If a pipe an inch in diameter will empty a cistern in an hour, how long will it take this pipe to empty the cistern if there is another pipe one third of an inch in diameter through which the fluid runs in?

Ans. $67\frac{1}{2}$ minutes.

14. If a pipe 3 inches in diameter will empty a cistern in 3 hours, how long will it take the pipe to empty the cistern if there are 3 other pipes each an inch in diameter through which the fluid runs in.

Ans. $4\frac{1}{2}$ hours.

EXERCISES.

The following Theorems, depending for their demonstration upon those already demonstrated, are introduced as exercises for the pupil. In some of them references are made to the propositions upon which the demonstration depends. They are not connected with the propositions in the following books, and can be omitted if thought best.

40. Every diameter bisects the circle and the circumference.

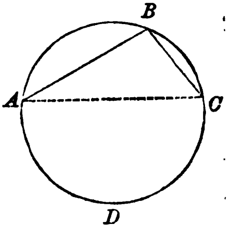
41. A straight line can meet the circumference of a circle in only two points. (4.) (I. 51.)

42. The diameter is greater than any other chord of the circle.

43. In the same or equal circles, when the sum of the arcs is less than a circumference, the greater arc is subtended by the greater chord; and, conversely, the greater chord is subtended by the greater arc.

Draw *AC*. (21.) (I. 47.)

What is the case when the sum of the arcs is greater than a circumference?



44. Equal chords are equally distant from the centre; and of two unequal chords the greater is nearer the centre.

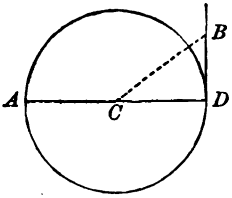
45. The shortest and the longest line that can be drawn from any point to a given circumference lies on the line that passes from the point to the centre of the circle.

46. Two parallels cutting the circumference of a circle intercept equal arcs.

47. A straight line perpendicular to a diameter at its extremity is a tangent to the circumference.

Draw *CB*. (I. 51.)

48. The lines joining the extremities of two diameters are parallel.



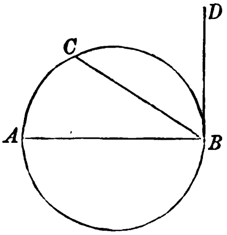
49. If the extremities of two chords are joined, the triangles thus formed are similar.

50. If two circumferences cut each other, the chord which joins their points of intersection is bisected at right angles by the line joining their centres. (17.)

51. If two circumferences touch each other, their centres and point of contact are in the same straight line, perpendicular to the tangent at the point of contact. (47.)

52. The distance between the centres of two circles whose circumferences cut one another, is less than the sum, but greater than the difference, of their radii.

53. Every angle inscribed in a segment greater than a semicircle is acute; and every angle inscribed in a segment less than a semicircle is obtuse. (21.)

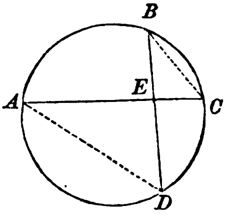


54. The angle made by a tangent and a chord is measured by half the included arc.

Draw the diameter AB. (47.) (21.)

55. The angle formed by two chords cutting each other within the circle is measured by half the sum of the intercepted arcs.

Join BC (in lower figure). (21.)



56. By moving the point of intersection of the two chords, show that (14) and (21) can be deduced from (55).

57. The segments of two chords cutting each other within a circle are reciprocally proportional.

Join AD, BC. (21.) (II. 20.)

58. The opposite angles of a quadrilateral inscribed in a circle are supplementary. (21.)

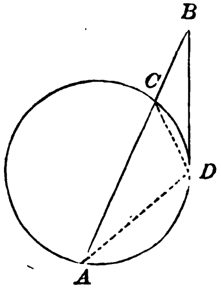
59. A quadrilateral whose opposite angles are supplementary, and no other, can be inscribed in a circle.

60. Circles are as the squares of their radii, or diameters, or circumferences. (32.)

61. The area of a sector is equal to half the product of its arc by the radius of the circle. (31.)

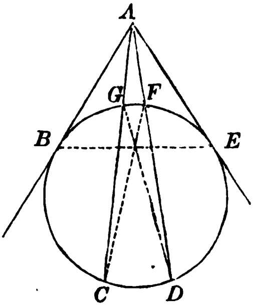
62. Show how to find the area of a segment of a circle.

63. The area of a circumscribed polygon is equal to half the product of its perimeter by the radius of the circle.



64. A tangent is a mean proportional between a secant drawn from the same point and the part of the secant without circle.

Join *A D*, *D C*. (54; 21.) (II. 57.)



65. The angle formed by two secants, two tangents, or a secant and a tangent cutting each other without the circle, is measured by half the difference of the intercepted arcs.

Join *C F*. (I. 39.) (21.)

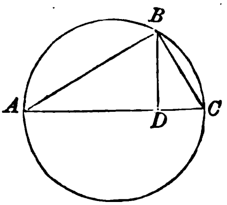
66. By moving the point of intersection, show that (21) can be deduced from (65). Show also that (46) can be deduced from (65).

67. Two secants drawn from the same point are to each other inversely as the parts of the secants without the circle.

Join *C F*, *D G*. (21.) (II. 57.)

68. Two tangents drawn to a circumference from the same point without this circumference are equal.

Join *B E*. Figure in (66.) (54.)



69. A perpendicular from a circumference to the diameter is a mean proportional between the segments of the diameter.

Join *A B*, *B C*. (23.) (II. 26.)

70. If from one end of a chord a diameter is drawn, and from the other end a perpendicular to this diameter, the chord is a mean proportional between the diameter and the adjacent segment of the diameter.

Join *A B*. (23.) (II. 25.)

71. The sum of the opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides. (68.)

BOOK IV.

GEOMETRY OF SPACE.

PLANES AND THEIR ANGLES.

DEFINITIONS.

1. A straight line is *perpendicular to a plane* when it is perpendicular to every straight line of the plane which it meets.

Conversely, the plane, in this case, is perpendicular to the line.

The *foot* of the perpendicular is the point in which it meets the plane.

2. A *line and a plane are parallel* when they cannot meet though produced indefinitely.

3. Two *planes are parallel* when they cannot meet though produced indefinitely.

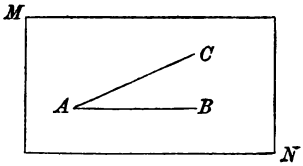
THEOREM I.

4. *A plane is determined,*

1st. *By a straight line and a point without that line;*

2d. *By three points not in the same straight line;*

3d. *By two intersecting straight lines.*



1st. Let the plane MN, passing through the line AB, turn upon this line as an axis until it contains the point C; the position of the plane is evidently determined; for if it is turned in either direction it will no longer contain the point C.

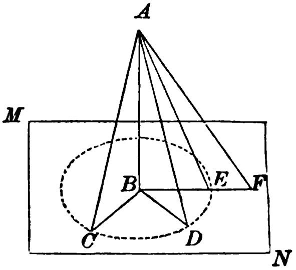
2d. If three points, A, B, C, not in the same straight line are given, any two of them, as A and B, may be joined by a straight line; then this is the same as the 1st case.

3d. If two intersecting lines AB, AC are given, any point, C, out of the line AB can be taken in the line AC; then the plane passing through the line AB and the point C contains the two lines AB and AC, and is determined by them.

5. *Corollary.* The intersection of two planes is a straight line; for the intersection cannot contain three points not in the same straight line, since only one plane can contain three such points.

THEOREM II.

6. *Oblique lines from a point to a plane equally distant from the perpendicular are equal; and of two oblique lines unequally distant from the perpendicular, the more remote is the greater.*



Let AC, AD be oblique lines drawn to the plane MN at equal distances from the perpendicular AB:

1st. $AC = AD$; for the triangles ABC, ABD are equal (I. 40).

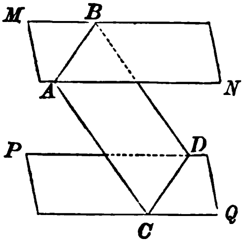
2d. Let AF be more remote. From BF cut off $BE = BD$ and draw AE; then $AF > AE$ (I. 51); and $AE = AD = AC$; therefore $AF > AD$ or AC.

7. *Cor.* 1. *Conversely,* equal oblique lines from a point to a plane are equally distant from the perpendicular; therefore they meet the plane in the circumference of a circle whose centre is the foot of the perpendicular. Of two unequal lines the greater is more remote from the perpendicular.

8. *Cor.* 2. The perpendicular is the shortest distance from a point to a plane.

THEOREM III.

9. *The intersections of two parallel planes with a third plane are parallel.*



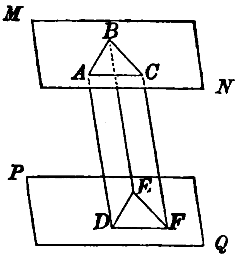
Let AB and CD be the intersections of the plane AD with the parallel planes MN and PQ; then AB and CD are parallel.

For the lines AB and CD cannot meet though produced indefinitely, since the planes MN and PQ in which they are cannot meet; and they are in the same plane AD; therefore they are parallel.

10. *Corollary.* Parallels intercepted between parallel planes are equal. For the opposite sides of the quadrilateral AD being parallel, the figure is a parallelogram; therefore $AC = BD$.

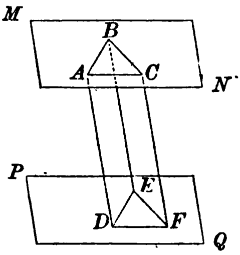
THEOREM IV.

11. *If two angles not in the same plane have their sides parallel and similarly situated, the angles are equal and their planes parallel.*



Let ABC and DEF be two angles in the planes MN and PQ, having their sides AB, BC respectively parallel to DE, EF, and similarly situated; then

1st. The angles ABC and DEF are equal. For, taking $ED = BA$, and $EF = BC$, and drawing AC, DF, AD, BE, and CF, the quadrilaterals AE and BF are parallelograms, since AB and BC are respectively equal and parallel to DE and EF; therefore AD and CF, being each equal and parallel to BE, are equal and parallel to

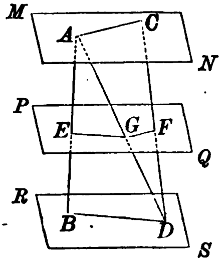


each other; and therefore $A F$ is a parallelogram, and $A C$ is equal to $D F$; therefore the two triangles $A B C$ and $D E F$, being mutually equilateral, are mutually equiangular, and the angles $A B C$ and $D E F$ are equal.

2d. The planes of these angles are parallel. For, since two intersecting lines determine a plane, the plane of the lines $A B$ and $B C$ must be parallel to the plane of the lines DE and EF, as $A B$ and $B C$ are respectively parallel to $D E$ and $E F$.

THEOREM V.

12. *If two straight lines are cut by parallel planes, they are divided proportionally.*



Let $A B$ and $C D$ be cut by the parallel planes MN, $P Q$, and $R S$, in the points A, E, B, and C, F, D; then

$$A E : EB = C F : F D$$

For, drawing $A D$ meeting the plane $P Q$ in G, the plane of the lines $A B$ and $A D$ cuts the parallel planes $P Q$ and $R S$ in $E G$ and $B D$; therefore $E G$ and $B D$ are parallel (9), and we have (II. 16)

$$A E : EB = AG : GD$$

The plane of the lines $A D$ and $C D$ cuts the parallel planes MN and PQ in $A C$ and GF; therefore $A C$ is parallel to $G F$; and we have

$$A G : GD = C F : F D$$

Hence we have (Pn. 11)

$$A E : EB = C F : F D$$

EXERCISES.

The following Theorems, depending for their demonstration upon those already demonstrated, are introduced as exercises for the pupil. In some of them references are made to the propositions upon which the demonstration depends. They are not connected with the propositions in the following books, and can be omitted if thought best.

13. An infinite number of planes can pass through a given line. (4.)

14. There can be but one perpendicular from a point to a plane.

15. A line perpendicular to each of two lines at their point of intersection is perpendicular to the plane of these lines. (4.) (I. 76.)

16. Parallel lines are equally inclined to the same plane.

17. State the converse of (16). Is it true?

18. Lines parallel to a line in a given plane are parallel to the plane.

19. State the converse of (18). Is it true?

20. Parallel planes are equally inclined to the same straight line.

21. State the converse of (20). Is it true?

22. Parallel lines included between parallel planes are equal.

BOOK V.

POLYEDRONS.

DEFINITIONS.

1. A **Polyedron** is a solid bounded by planes.

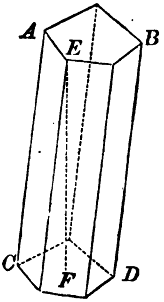
The bounding planes are called *faces;* their intersections, *edges;* the intersections of the edges, *vertices.*

2. The **Volume** of a solid is the measure of its magnitude. It is expressed in units which represent the number of times it contains the cubical unit taken as a standard.

3. **Equivalent Solids** are those which are equal in volume.

4. **Similar Solids** are those whose homologous lines have a constant ratio. (*Corollary.*) It follows that similar solids are bounded by the same number of similar polygons similarly situated.

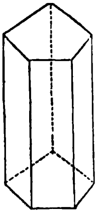
PRISMS AND CYLINDERS.



5. A **Prism** is a polyedron two of whose faces are equal polygons having their homologous sides parallel. (*Corollary.*) The other faces are parallelograms.

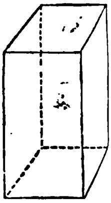
The equal parallel polygons are called *bases;* as *A B* and *C D.*

6. The **Altitude** of a prism is the perpendicular distance between its bases; as *E F.*



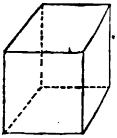
7. A **Right Prism** is one whose other faces are perpendicular to its bases. (*Corollary.*) Its lateral faces are rectangles.

8. A prism is called *triangular*, *quadrangular*, or *pentagonal*, according as its base is a triangle, a quadrangle, or a pentagon; and so on.

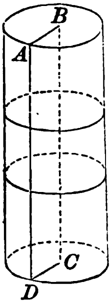


9. A **Parallelopiped** is a prism whose bases are parallelograms. (*Corollary.*) It follows that all its faces are parallelograms.

10. A **Right Parallelopiped** has all its faces rectangles.



11. A **Cube** is a parallelopiped whose faces are all squares. (*Corollary.*) It follows that its faces are all equal, and the parallelopiped right.

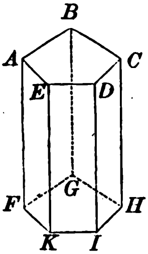


12. A **Cylinder** is a right prism whose parallel faces are regular polygons of an infinite number of sides, that is, whose parallel faces are circles. A cylinder can be described by the revolution of a rectangle about one of its sides which remains fixed. The side opposite the fixed side describes the *convex surface*, and the other two sides the two circular bases. Thus the rectangle $A\,B\,C\,D$ revolving about $B\,C$ would describe the cylinder, the side $A\,D$ the convex surface, and $A\,B$, $D\,C$ the circular bases.

13. The **Axis** of a cylinder is the straight line joining the centres of the two bases; or it is the fixed side of the rectangle whose revolution describes the cylinder; as $B\,C$.

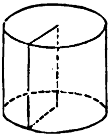
THEOREM I.

14. *The convex surface of a right prism is equal to the perimeter of its base multiplied by its altitude.*



Let AH be a right prism; its convex surface is equal to $FG + GH + HI + IK + KF$ multiplied by its altitude AF.

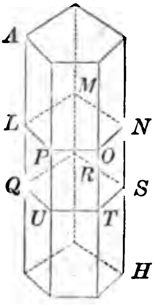
For the convex surface is equal to the sum of the rectangles AG, BH, CI, &c. The area of the rectangle $AG = FG \times AF$; the area of $BH = GH \times BG$; of $CI = HI \times CH$; and so on. But the edges AF, BG, CH, &c. are equal to each other and to the altitude of the prism; and the bases of these rectangles together form the perimeter of the prism. Therefore the sum of these rectangles, that is, the convex surface of the right prism, is equal to the perimeter of its base multiplied by its altitude.



15. *Corollary.* As a cylinder is a right prism (12), this demonstration includes the cylinder. If, then, $R =$ the radius of the base, and $A =$ the altitude of a cylinder, the convex surface $= 2\pi RA$.

THEOREM II.

16. *The sections of a prism made by parallel planes are equal polygons.*



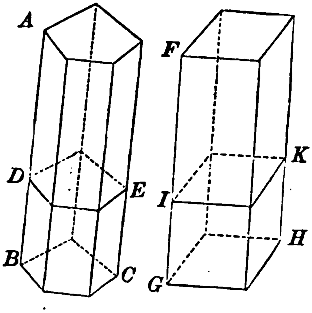
Let the prism AH be intersected by the parallel planes LN and QS; then LN and QS are equal polygons. For LM, MN, NO, &c. are respectively parallel to QR, RS, ST, &c. (IV. 9), and similarly situated; therefore the angles L, M, N, O, P are respectively equal to the angles Q, R, S, T, U (IV. 11); and the polygons LN and QS are mutually equiangular. Also the sides LM, MN, NO, &c. are respectively equal to QR, RS, ST, &c. (I. 62). Therefore the polygons, being mutually equiangular and equilateral, are equal (II. 6).

17. *Cor.* 1. A section made by a plane parallel to the base is equal to the base.

18. *Cor.* 2. A section of a cylinder made by a plane parallel to the base is a circle equal to the base.

THEOREM III.

19. *Prisms having equivalent bases and equal altitudes are equivalent.*



Let AC and FH be two prisms having equal altitudes and their bases BC, GH equivalent; the prisms are equivalent.

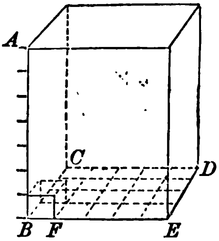
Let DE and IK be sections made by planes respectively parallel to the bases BC and GH; these sections are respectively equal to the bases (17); therefore the section DE is equivalent to IK, at whatever distance from the base either may be. If, therefore, the planes of these sections move, remaining always parallel to the bases, as the sections will always be equivalent, it is evident that in moving over an equal length of altitude the sections will move over equal volumes; therefore, as the altitudes are equal, the prisms are equivalent.

20. *Corollary.* Any prism is therefore equivalent to a right prism having an equivalent base and an equal altitude.

THEOREM IV.

21. *The volume of a right parallelopiped is equal to the product of its three dimensions.*

Let $A\,D$ be the right parallelopiped; then its volume is equal to $B\,C \times B\,E \times B\,A$. Suppose BF, the linear unit, is contained in $B\,C$ four times, in BE five times, and in $B\,A$ seven times; then dividing $B\,C$, $B\,E$, $B\,A$ respectively into four, five, and seven equal parts, and passing planes through the several points of division parallel to the sides of the parallelopiped, there will be formed a number of cubes equal to each other (19), and each equal to the cube whose edge is the linear unit. It is evident also that the whole number of cubes is equal to the product of the three dimensions, or $4 \times 5 \times 7 = 140$. This demonstration is applicable, whatever the number of units in the linear dimensions may be. Therefore the volume of a right parallelopiped is equal to the product of its three dimensions.



22. *Scholium.* If the three dimensions are incommensurable, the linear unit can be taken infinitely small, that is, so small that the remainder will be infinitesimal and can be neglected.

23. *Cor.* 1. As the base is equal to $B\,C \times B\,E$, the volume of a right parallelopiped is equal to the product of its base by its altitude.

24. *Cor.* 2. The volume of a cube is equal to the cube of its edge.

THEOREM V.

25. *The volume of any prism is equal to the product of its base by its altitude.*

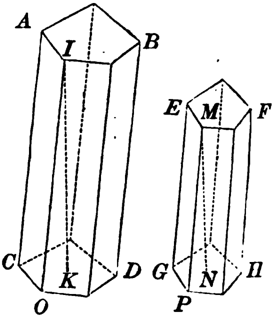
For any prism is equivalent to a right parallelopiped, having an equivalent base and the same altitude (20); and the volume of the equivalent right parallelopiped is equal to the product of its base by its altitude; therefore the volume of any prism is equal to the product of its base by its altitude.

26. *Corollary.* As a cylinder is a right prism, this demonstration includes the cylinder. If, therefore, $R =$ the radius of base, $A =$ the altitude, and $V =$ the volume of a cylinder,

$$V = \pi R^2 A = \tfrac{1}{4}\pi D^2 A$$

THEOREM VI.

27. *Similar prisms are as the cubes of their homologous lines.*



Let AD and EH be similar prisms whose altitudes are IK and MN. Let V represent the volume of AD, and v the volume of EH; then

$$V : v = IK^3 : MN^3 = AC^3 : EG^3$$
$$= CO^3 : GP^3$$

For (25) $V = CD \times IK$ and $v = GH \times MN$, therefore

$$V : v = CD \times IK : GH \times MN$$

But (II. 31) $CD : GH = CO^2 : GP^2$

and (4) $IK : MN = CO : GP$

Multiplying the last two proportions together we have

$$CD \times IK : GH \times MN = CO^3 : GP^3$$

therefore (Pn. 11) $V : v = CO^3 : GP^3$

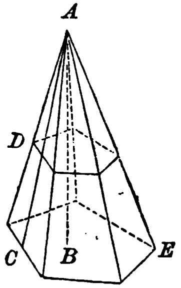
But in similar solids homologous lines have a constant ratio (4); therefore $V : v$ as the cubes of any homologous lines.

PYRAMIDS AND CONES.

DEFINITIONS.

28. A **Pyramid** is a polyedron bounded by a polygon called the base, and by triangular planes meeting at a common point called the vertex.

29. A pyramid is called *triangular*, *quadrangular*, *pentagonal*, according as its base is a triangle, a quadrangle, or a pentagon; and so on.

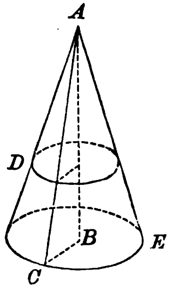


30. The **Altitude** of a pyramid is the perpendicular distance from its vertex to its base; as $A\ B$.

31. A **Right Pyramid** is one whose base is a regular polygon and in which the perpendicular from the vertex passes through the centre of the base.

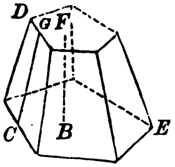
32. The **Slant Height** of a right pyramid is the perpendicular distance from the vertex to the base of any one of its lateral faces; as $A\ C$.

33. A **Cone** is a right pyramid whose base is a regular polygon of an infinite number of sides, that is, whose base is a circle. A cone can be described by the revolution of a right triangle about one of its sides which remains fixed. The other side describes the circular base, and the hypothenuse the *convex surface*. Thus the right triangle $A\ B\ C$ revolving about $A\ B$ would describe the cone, $B\ C$ the base, and the hypothenuse $A\ C$ the convex surface.



34. The **Axis** of a cone is the line from the vertex to the centre of the base; or it is the fixed side of the right triangle whose revolution describes the cone; as $A\ B$.

35. *Corollary.* The axis of a cone is perpendicular to the base, and is therefore the *altitude* of the cone.



36. A **Frustum** of a pyramid is a part of the pyramid included between the base and a plane cutting the pyramid parallel to the base; as DE.

37. The **Altitude** of a frustum is the perpendicular distance between the two parallel planes or bases; as FB.

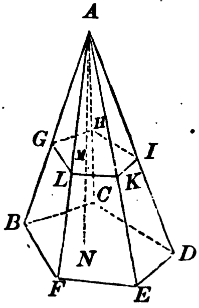
38. The **Slant Height** of a frustum of a right pyramid is the perpendicular distance between the parallel edges of the bases; as GC.

THEOREM VII.

39. *If a pyramid is cut by a plane parallel to its base,*

1st. *The edges and altitude are divided proportionally;*

2d. *The section is a polygon similar to the base.*



Let $A-BCDEF$ be a pyramid whose altitude is AN, cut by a plane GI parallel to the base; then

1st. The edges and the altitude are divided proportionally.

For suppose a plane passed through the vertex A parallel to the base; then the edges and altitude, being cut by three parallel planes, are divided proportionally (IV. 12), and we have

$$AB : AG = AC : AH = AD : AI = AN : AM$$

2d. The section GI is similar to the base BD.

For the sides of GI are respectively parallel to the sides of BD (IV. 9), and similarly situated; therefore the polygons GI, BD are mutually equiangular. Also, as GL is parallel to BF, and LK to FE, the triangles ABF and AGL are similar, and the triangles AFE and ALK; therefore

$$GL : BF = AL : AF, \text{ and } LK : FE = AL : AF$$

Therefore $$GL : BF = LK : FE$$

In the same manner we should find

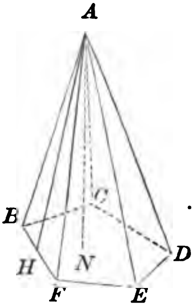
$$LK : FE = KI : ED = IH : DC, \text{ \&c.}$$

Therefore the polygons GI and BD are similar (II. 19).

40. *Corollary.* A section of a cone made by a plane parallel to the base is a circle.

THEOREM VIII.

41. *The convex surface of a right pyramid is equal to the perimeter of its base multiplied by half its slant height.*



Let A–$BCDEF$ be a right pyramid whose slant height is AH; its convex surface is equal to $BC + CD + DE + EF + FB$ multiplied by half of AH.

The edges AB, AC, AD, AE, AF, being equally distant from the perpendicular AN (II. 34), are equal (IV. 6); and the bases BC, CD, DE, &c. are equal; therefore the isosceles triangles ABC, ACD, ADE, &c. are all equal (I. 48); and their altitudes are equal. The area of ABC is $BC \times \frac{1}{2} AH$ (II. 11); of ACD is $CD \times \frac{1}{2} AH$; and so on. Therefore the sum of the areas of these triangles, that is, the convex surface of the right pyramid, is $(BC + CD + DE + EF + FB) \frac{1}{2} AH$.

42. *Corollary.* As a cone is a right pyramid (33), this demonstration includes the cone. If, therefore, $R =$ the radius of the base, and $S =$ the slant height of a cone,

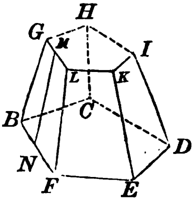
$$\text{its convex surface} = 2 \pi R \tfrac{1}{2} S = \pi R S$$

If a plane parallel to the base and bisecting the altitude be drawn, as the section will be a circle (40) with a radius and circumference one half the radius and circumference of the base, therefore, if $r' =$ the radius of this section,

$$\text{the convex surface} = 2\pi r' S$$

THEOREM IX.

43. *The convex surface of a frustum of a right pyramid is equal to the sum of the perimeter of its two bases multiplied by half its slant height.*



Let GD be the frustum of a right pyramid; its convex surface is equal to $GH + HI + IK + KL + LG + BC + CD + DE + EF + FB$ multiplied by half MN.

The lateral faces of a frustum of a right pyramid are equal trapezoids (39; II. 6); and their altitudes are all equal. The area of GC (II. 14) is $(GH + BC) \times \frac{1}{2} MN$; of HD is $(HI + CD) \times \frac{1}{2} MN$; and so on. Therefore the sum of the areas of these trapezoids, that is, the convex surface of the frustum of the right pyramid, is $GH + HI + IK + KL + LG + BC + CD + DE + EF + FB$ multiplied by half MN.

44. *Cor.* 1. If the frustum is cut by a plane parallel to its two bases, and at equal distances from each base, this plane will bisect the edges GB, HC, ID, &c. (39); and the area of each trapezoid is equal to its altitude multiplied by the line joining the middle points of the sides which are not parallel (II. 15). Therefore the convex surface of a frustum of a right pyramid is equal to the perimeter of a section midway between the bases multiplied by its slant height.

45. *Cor.* 2. As a cone is a right pyramid (33), this demonstration includes the frustum of a cone. If, therefore, R and

$r =$ the radii of the two bases of the frustum of a right cone, and $S =$ its slant height,

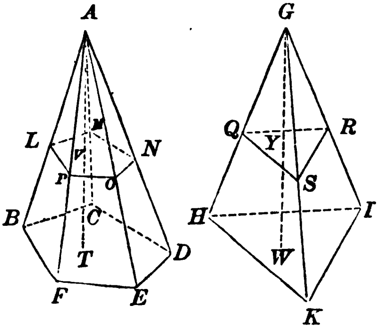
$$\text{its convex surface} = (2\pi R + 2\pi r)\tfrac{1}{2}S = (\pi R + \pi r)S$$

If $r' =$ the radius of a section midway between and parallel to the bases,

$$\text{the convex surface} = 2\pi r' S$$

THEOREM X.

46. *If two pyramids having equal altitudes are cut by planes parallel to their bases and at equal distances from their vertices, the sections are to each other as their bases.*



Let $A-BCDEF$ and $G-HIK$ be two pyramids of equal altitudes AT, GW, cut by the planes $LMNOP$ and QRS parallel respectively to the bases and at equal distances from the vertices A and G, then

$$LMNOP : QRS = BCDEF : HIK$$

For as the polygons $LMNOP$ and $BCDEF$ are similar (39)

$$LMNOP : BCDEF = \overline{LP}^2 : \overline{BF}^2 = \overline{AL}^2 : \overline{AB}^2 = \overline{AV}^2 : \overline{AT}^2$$

In like manner

$$QRS : HIK = \overline{GY}^2 : \overline{GW}^2$$

But as $AV = GY$ and $AT = GW$

therefore

$$LMNOP : BCDEF = QRS : HIK$$

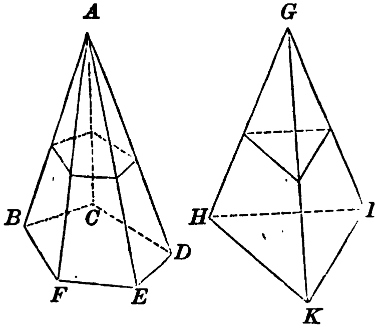
or (Pn. 15)

$$LMNOP : QRS = BCDEF : HIK$$

47. *Corollary.* If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to their bases and at equal distances from their vertices are equivalent.

THEOREM XI.

48. *Pyramids having equivalent bases and the same altitude are equivalent.*

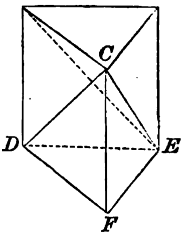


Let $A-BCDEF$ and $G-HIK$ be pyramids having equivalent bases and equal altitudes; then the two pyramids are equivalent.

For, if at equal distances from the vertex sections are formed by planes parallel respectively to their bases, these sections are equivalent (47). If now the planes forming these sections be supposed to move, remaining always parallel to the bases, and each keeping the same distance from the vertex as the other, these sections, always being equivalent to each other, will move over equal volumes; therefore, as the altitudes are equal, the pyramids must be equivalent.

THEOREM XII.

49. *A triangular pyramid is one third of a triangular prism of the same base and altitude.*



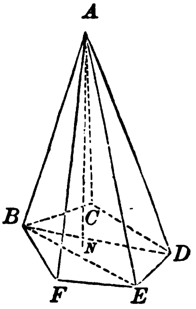
Let $C-DEF$ be a triangular pyramid and $ABC-DEF$ be a triangular prism on the same base DEF; then $C-DEF$ is one third of $ABC-DEF$.

Taking away the pyramid $C-DEF$ there remains the quadrangular pyramid whose vertex is C and base the parallelogram $ABED$. Through the points A, C, E pass a plane; it will divide the pyramid $C-ABED$ into two triangular pyramids, which are equivalent to each other (48), since their bases are halves of the parallelogram $ABED$, and they have the same altitude, the perpendicular from their vertex C to the base $ABED$. But the pyramid $C-ABE$, that is, $E-ABC$, is equivalent to the pyramid $C-DEF$, as they have equal bases ABC and DEF, and the same altitude (48). Therefore the three pyramids are equivalent and the given pyramid is one third of the prism.

50. *Corollary.* The volume of a triangular pyramid is equal to one third the product of its base by its altitude.

THEOREM XIII.

51. *The volume of any pyramid is equal to one third of the product of its base by its altitude.*



Let $A-BCDEF$ be any pyramid; its volume is equal to one third the product of its base $BCDEF$ by its altitude AN.

Planes passing through the vertex A and the diagonals of the base BD, BE, will divide the pyramid into triangular pyramids whose bases together compose the base of the given pyramid and which have as their common altitude AN, the altitude of the given pyramid. The volume of the given pyramid is equal to the sum of the volumes of the several triangular pyramids, which is equal to one third of the sum of their bases multiplied by their common altitude; that is, is equal to one third of the product of the base $BCDFE$ by the altitude AN.

52. *Cor.* 1. As a cone is a right pyramid (33), this demonstration includes the cone. A cone, therefore, is one third of a cylinder, or of any pyramid, of equivalent base and the same altitude. If $R =$ radius of the base, $A =$ the altitude, and $V =$ the volume of a cone, $V = \frac{1}{3} \pi R^2 A$.

53. *Cor.* 2. The ratio of similar pyramids to one another is the same as that of similar prisms; that is, as the cubes of homologous lines.

THE SPHERE.

DEFINITIONS.

54. A **Sphere** is a solid bounded by a curved surface, of which every point is equally distant from a point within called the *centre.* A sphere can be described by the revolution of a semicircle about its diameter, which remains fixed.

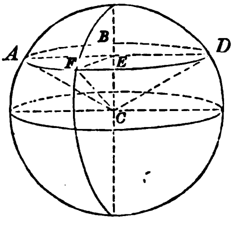
55. The **Radius** of a sphere is the straight line from the centre to any point of the surface.

56. The **Diameter** of a sphere is a straight line passing through the centre and terminating at either end at the surface.

57. *Corollary.* All the radii of a sphere are equal; all the diameters are equal, and each is double the radius.

THEOREM XIV.

58. *Every section of a sphere made by a plane is a circle.*



Let ABD be a section made by a plane cutting the sphere whose centre is C; then is ABD a circle.

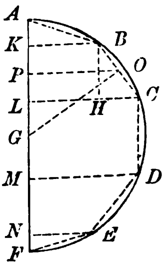
Draw CE perpendicular to the plane, and to the points A, D, F, where the plane cuts the surface of the sphere, draw CA, CD, CF. As CA, CD, CF are radii of the sphere they are equal, and are therefore equally distant from the foot of the perpendicular CE (IV. 7). Therefore EA, ED, EF are equal, and the section ABD is a circle whose centre is E.

59. *Corollary.* If the section passes through the centre of the sphere, its radius will be the radius of the sphere.

60. *Definition.* A section made by a plane passing through the centre of a sphere is called a *great circle.* A section made by a plane not passing through the centre is called a *small circle.*

THEOREM XV.

61. *The surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*



Let $ABCDEF$ be the semicircle by whose revolution about the diameter AF, the sphere may be described; then the surface of the sphere is equal to the diameter AF multiplied by the circumference of the circle whose radius is GA, or $= AF \times$ circ. GA.

Let $ABCDEF$ be a regular semi-decagon inscribed in the semicircle. Draw GO perpendicular to one of its sides, as BC.

Draw BK, OP, CL, DM, EN perpendicular to the diameter AF, and BH perpendicular to CL. The surface described by BC is the convex surface of the frustum of a cone, and is equal to $BC \times$ circ. PO (45). But the triangles BCH and POG are similar (II. 21); therefore

$$BC : BH \text{ or } KL = GO : PO$$

or (III. 28) $$BC : KL = \text{circ. } GO : \text{circ. } PO$$

$$\therefore BC \times \text{circ. } PO = KL \times \text{circ. } GO$$

That is, the surface described by BC is equal to the altitude KL multiplied by circ. GO, or the radius of the circle inscribed in the polygon. In like manner it can be proved that the surfaces described by AB, CD, DE, and EF are respectively equal to their altitudes AK, LM, MN, and NF multiplied by circ. GO. Therefore the entire surface described by the semi-polygon will be equal to

$$(AK + KL + LM + MN + NF)\text{ circ. } GO = AF \times \text{circ. } GO$$

This demonstration is true, whatever the number of sides of the semi-polygon; it is true, therefore, if the number of sides is infinite, in which case the semi-polygon would coincide with the semicircle; and the surface described by the semi-polygon would be the surface of the sphere, and the radius of the inscribed polygon would be the radius of the sphere. Therefore we have the surface of the sphere equal to

$$AF \times \text{circ. } GA$$

62. *Corollary.* Let $S =$ the surface of the sphere, $C =$ the circumference, $R =$ the radius, $D =$ the diameter, then we have (III. 30) $C = 2\pi R$, or πD

Therefore $S = 2\pi R \times 2R = 4\pi R^2$, or πD^2

That is, *the surface of a sphere is equal to the square of its diameter multiplied by* 3.14159.

THEOREM XVI.

63. *The volume of a sphere is the product of its surface by one third of its radius.*

A sphere may be conceived to be composed of an infinite number of pyramids whose vertices are at the centre of the sphere, and whose bases, being infinitely small planes, coincide with the surface of the sphere. The altitude of each of these pyramids is the radius of the sphere, and the sum of the surfaces of their bases is the surface of the sphere. The volume of each pyramid is the product of the area of its base by one third of its altitude, that is, of the radius of the sphere (51); and the volume of all the pyramids, that is, of the sphere, is, therefore, the product of the surface of the sphere by one third of its radius.

64. *Cor.* 1. Let $V =$ the volume of the sphere, and R, D, and S the same as in (62). Then, as (62)

$$S = 4\pi R^2, \text{ or } \pi D^2$$

$$V = 4\pi R^2 \times \tfrac{1}{3} R = \tfrac{4}{3}\pi R^3, \text{ or } \tfrac{1}{6}\pi D^3$$

That is, *the volume of a sphere is the cube of the diameter multiplied by* .5235.

65. *Cor.* 2. As in these equations $\frac{4}{3}\pi$ and $\frac{1}{6}\pi$ are constant, *the volumes of spheres vary as the cubes of their radii, or as the cubes of their diameters.*

PRACTICAL QUESTIONS.

1. How many square feet in the convex surface of a right prism whose altitude is 2 feet, and whose base is a regular hexagon of which each side is 8 inches long? How many square feet in the whole surface?

2. The radius of the base of a cylinder is 6 inches, and its altitude 3 feet; how many square feet in the whole surface?

3. What is the number of feet in the bounding planes of a cube whose edge is 5 feet? The number of solid feet in the cube?

4. What is the number of feet in the bounding planes of a right parallelopiped whose three dimensions are 4, 7, and 9 feet? The number of cubic feet in the parallelopiped?

5. What is the number of cubic feet in the right prism whose dimensions are given in the first example?

6. What is the number of cubic feet in the cylinder whose dimensions are given in the second example?

7. The altitude of a prism is 9 feet and the perimeter of the base 6 feet. What is the altitude and perimeter of the base of a similar prism one third as great?

8. What is the ratio of the volumes of two cylinders whose altitudes are as 3 : 6?

9. How many square feet in the convex surface of a right pyramid whose slant height is 3 feet, and whose base is a regular octagon of which each side is 2 feet long?

10. How many square feet in the convex surface of a cone whose slant height is 5 feet and whose base has a radius of 2 feet? How many square feet in the whole surface?

11. How many cubic feet in a right quadrangular pyramid whose altitude is 10 feet, and whose base is 3 feet square?

12. How many cubic feet in the cone whose dimensions are given in the tenth example?

13. The slant height of a frustum of a right pyramid is 6 feet, and the perimeters of the two bases are 18 feet and 12 feet respectively; what is the convex surface of the frustum?

14. What would be the slant height of the pyramid whose frustum is given in the preceding example?

15. What is the whole surface of a frustum of a cone whose altitude is 8 feet, and of whose bases the radii are 11 feet and 5 feet respectively?

16. The altitude of a pyramid is 25 feet, and its base is a rectangle 8 feet by 6 ; how many cubic feet in the pyramid ?

17. The altitude of a cone is 20 feet, and the radius of its base 5 feet ; how many cubic feet in the cone ?

18. How many cubic feet in a frustum of the cone given in the preceding example, cut off by a plane 5 feet from the base ?

19. How far from the base must a cone whose altitude is 12 feet be cut off so that the frustum shall be equivalent to one half of the cone ?

20. How many square feet in the surface of a sphere whose radius is 6 feet ?

21. How many cubic feet in a sphere whose radius is 8 feet ?

22. What is the ratio of the volumes of two spheres whose radii are as 4 : 8 ?

23. Are spheres always similar solids ? Are cones ?

24. What is the least number of planes that can enclose a space ?

EXERCISES.

66. The convex surfaces of prisms or pyramids of equal altitudes are as the perimeters of their bases. (14.)

67. The opposite faces of a parallelopiped are equal and parallel.

68. The four diagonals of a parallelopiped bisect each other.

69. A plane passing through the opposite edges of a parallelopiped bisects the parallelopiped.

70. In a right parallelopiped the diagonals are equal; and the square of each is equal to the sum of the squares of the three dimensions.

71. In a cube the square of a diagonal is three times the square of an edge.

72. Prisms are to each other as the products of their bases by their altitudes. (25.)

73. Prisms with equivalent bases are as their altitudes; with equal altitudes, as their bases. (72.)

74. Polygons formed by parallel planes cutting a pyramid are as the squares of their distances from the vertex. (39; II. 31.)

75. Pyramids are to each other as the products of their bases by their altitudes. (51.)

76. Pyramids with equivalent bases are as their altitudes; with equal altitudes, as their bases. (75.)

77. How can Theorem VIII. be proved from Theorem IX.?

78. If a pyramid is cut by a plane parallel to its base, the pyramid cut off will be similar to the whole pyramid. (39; 4).

79. In a sphere great circles bisect each other.

80. A great circle bisects a sphere. (54.)

81. The centre of a small circle is in the perpendicular from the centre of the sphere to the small circle.

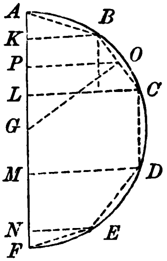
82. Small circles equally distant from the centre of a sphere are equal.

83. The intersection of the surfaces of two spheres is the circumference of a circle.

84. The arc of a great circle can be made to pass through any two points on the surface of a sphere. (IV. 4.)

85. *Definition.* A plane is tangent to a sphere when it touches but does not cut the sphere.

86. Prove that the radius of a sphere to the point of tangency of a plane is perpendicular to the plane. (IV. 8.)



87. As the semi-decagon revolves about $A F$, what kind of a solid is described by the triangle ABK? What by the trapezoid KC? By LD?

88. The surface described by the line $AB = AK \times$ circ. GO.

Draw from G a perpendicular to AB, and from the point where it meets AB a perpendicular to AF. (42.)

89. The surface described by the line $CD = LM \times$ circ. GO. (15.)

90. *Definition.* The surfaces described by the arcs AB, BC, CD, &c. are called *zones.*

91. The area of a zone is equal to the product of its altitude by the circumference of a great circle.

92. Zones on the same or equal spheres are as their altitudes.

93. The surface of a sphere is four times the surface of one of its great circles. (62; III. 32.)

94. *Definition.* A polyedron is circumscribed about a sphere when its faces are each tangents to the sphere. In this case the sphere is inscribed in the polyedron.

95. The surface of a sphere is equal to the convex surface of the circumscribed cylinder. (62; 15.)

96. *Definition.* A **Spherical Sector** is the solid described by any sector of a semicircle as the semicircle revolves about its diameter.

97. The volume of a spherical sector is equal to the product of the surface of the zone forming its base by one third of the radius of the sphere of which it is a part.

98. A **Spherical Segment** is a part of a sphere included by two parallel planes cutting or touching the sphere. When one plane touches and one cuts the sphere, the spherical segment is called *a spherical segment of one base;* when both cut, *a spherical segment of two bases.*

99. How can the volume of a spherical segment of one base be found? A spherical segment of two bases?

100. A sphere is two thirds of the circumscribed cylinder.

101. A cone, hemisphere, and cylinder having equal bases and the same altitude are as the numbers 1, 2, 3.

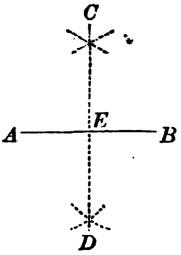
BOOK VI.

PROBLEMS OF CONSTRUCTION.

In the preceding demonstrations we have assumed that our figures were already constructed. The Problems of Construction given in this Book depend for their solution upon the principles of the preceding Books. In some of the problems the construction and demonstration are given in full; in others the construction is given and the propositions necessary to prove the construction referred to in the order in which they are to be used, and the pupil must complete the demonstration. In a few instances references are made to the Exercises appended to the previous Books. In such cases either the propositions to which reference is made can be demonstrated or the problem omitted.

PROBLEM I.

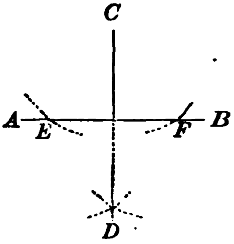
1. *To bisect a given straight line.*



Let $A B$ be the given straight line. From A and B as centres with a radius greater than half of $A B$, describe arcs cutting one another at C and D; join C and D cutting $A B$ at E, and the line $A B$ is bisected at E. For C and D being each equally distant from A and B, the line $C D$ must be perpendicular to $A B$ at its middle point (converse of I. 53).

PROBLEM II.

2. *From a given point without a straight line to draw a perpendicular to that line.*

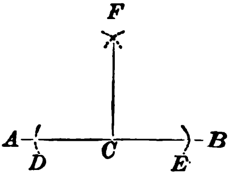


Let C be the point and $A\,B$ the line.

From C as a centre describe an arc cutting $A\,B$ in two points E and F; with E and F as centres, with a radius greater than half $E\,F$, describe arcs intersecting at D. Draw $C\,D$, and it is the perpendicular required (converse of I. 53).

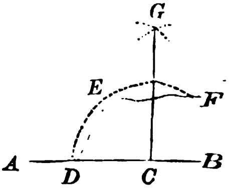
PROBLEM III.

3. *From a given point in a straight line to erect a perpendicular to that line.*



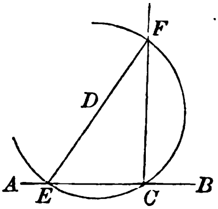
Let C be the given point and $A\,B$ the given line.

With C as a centre describe an arc cutting $A\,B$ in D and E; with D and E as centres, with a radius greater than $D\,C$, describe arcs intersecting at F. Draw $C\,F$, and it is the perpendicular required (converse of I. 53).



Second Method. With C as a centre describe an arc $D\,E\,F$; take the distances $D\,E$ and $E\,F$ equal to $C\,D$, and from E and F as centres, with a radius greater than half the distance from E to F, describe arcs intersecting at G. Draw $C\,G$, and it is the perpendicular required (III. 33; III. 16; III. 15).

Third Method. With any point, D, without the line $A\,B$, with a radius equal to the distance from D to C, describe an arc cutting $A\,B$ at E; draw the diameter $E\,D\,F$. Draw $C\,F$, and it is the perpendicular required (III. 23).

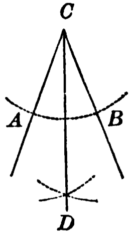


PROBLEM IV.

4. *To bisect a given arc, or angle.*

1st. Let $A\,B$ be the given arc. Draw the chord $A\,B$ and bisect it with a perpendicular (1; III. 16).

2d. Let C be the given angle.

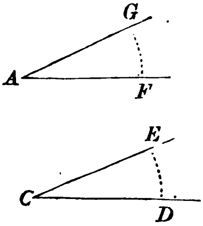


With C as a centre describe an arc cutting the sides of the angle in A and B; bisect the arc $A\,B$ with the line $C\,D$, and it will also bisect the angle C (III. 11).

PROBLEM V.

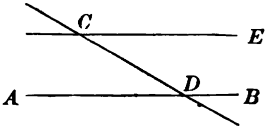
5. *At a given point in a straight line to make an angle equal to a given angle.*

Let A be the given point in the line $A\,B$, and C the given angle. With C as a centre describe an arc $D\,E$ cutting the sides of the angle C; with A as a centre, with the same radius, describe an arc; with F as a centre, with a radius equal to the distance from D to E, describe an arc cutting the arc $F\,G$. Draw $A\,G$. The angle $A = C$ (III. 12; III. 11).



PROBLEM VI.

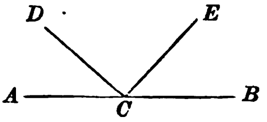
6. *Through a given point to draw a line parallel to a given straight line.*



Let C be the given point, and $A\,B$ the given line. From C draw a line $C\,D$ to $A\,B$; at C in the line $D\,C$ make an angle $D\,C\,E$ equal to $C\,D\,A$ (5); $C\,E$ is parallel to $A\,B$ (I. 18).

PROBLEM VII.

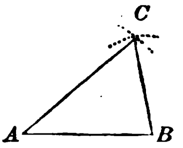
7. *Two angles of a triangle given, to find the third.*



Draw an indefinite line $A B$; at any point C make an angle $A\,C\,D$ equal to one of the given angles, and $D\,C\,E$ equal to the other (5). Then $E\,C\,B$ is the third angle (I. 7; I. 33).

PROBLEM VIII.

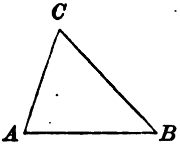
8. *The three sides of a triangle given, to construct the triangle.*



Take $A\,B$ equal to one of the given sides; with A as a centre, with a radius equal to another of the given sides, describe an arc, and with B as a centre, with a radius equal to the remaining side, describe an arc intersecting the first arc at C. Draw $A\,C$ and $C\,B$, and $A\,C\,B$ is evidently the triangle required.

PROBLEM IX.

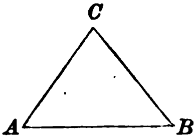
9. *Two sides and the included angle of a triangle given, to construct the triangle.*



Draw $A B$ equal to one of the given sides; at B make the angle $A B C$ equal to the given angle (5), and take $B C$ equal to the other given side; join A and C, and $A B C$ is evidently the triangle required.

PROBLEM X.

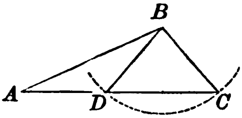
10. *Two angles and a side of a triangle given, to construct the triangle.*



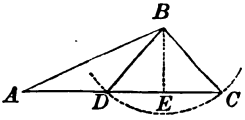
If the angles given are not both adjacent to the given side, find the third angle by (7). Then draw $A B$ equal to the given side, and at B make an angle $A B C$ equal to one of the angles adjacent to $A B$, and at A make an angle $B A C$ equal to the other angle adjacent to $A B$, and $A B C$ is evidently the triangle required.

PROBLEM XI.

11. *Two sides of a triangle and the angle opposite one of them given, to construct the triangle.*



Draw an indefinite line $A C$; at A make the angle $C A B$ equal to the given angle, and take $A B$ equal to the side adjacent to the given angle; with B as a centre, with a radius equal to the other given side, describe an arc cutting $A C$. If the given angle A is acute, 1st. The given side BC, opposite the given angle, may be less than the other given side; then the arc described from B as a centre will cut AC in two points, C and D, on the same side of A, and, drawing BC and BD, the triangles ABC and ABD (whose angle BDA is the supplement of the angle BCA), both satisfy the given conditions.



2d. The given side opposite the given angle may be equal to the perpendicular BE; then the arc described from B as a centre will touch AC, and the right triangle ABE is the only one that can satisfy the given conditions.

3d. The side opposite the given angle may be greater than the other given side; then the arc described from B as a centre AC will cut ~~AB~~ in C, and in another point on the other side of A. In this case there can be but one triangle ABC satisfying the given conditions, the triangle formed on the opposite side of AB containing not the given angle but its supplement.

4th. If the given angle is obtuse, the given side opposite the given angle must be greater than the other given side, and as in the last case above there can be but one solution.

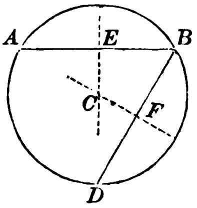
12. *Scholium.* If the side opposite the given angle A is less than the perpendicular, or if the given angle is right or obtuse, and at the same time the side opposite the given angle is less than the other given side, the solution is impossible.

13. *Corollary.* From this and the preceding Problem and Theorems VIII., IX., and XIV. of Book I., it follows that with the exception of the ambiguity pointed out in the first part of this Problem, two triangles are equal if any three parts, of which one is a side, of the one are equal to the corresponding parts of the other.

PROBLEM XII.

14. *To find the centre of a given circumference or of a given arc.*

Let $A B D$ be the given circumference, or arc.



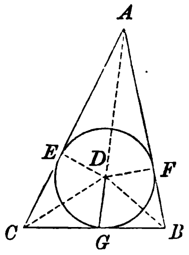
Draw any two chords not parallel to each other, as $A B$, $B D$, and bisect these chords by the perpendiculars $C E$ and $C F$. These perpendiculars will intersect at the centre of the circumference or arc (III. 17).

15. *Scholium.* By the same construction a circumference may be made to pass through any three given points; or a circle circumscribed about a given triangle.

PROBLEM XIII.

16. *To inscribe a circle in a given triangle.*

Let $A B C$ be the given triangle.



Bisect any two of its angles. With the point D, where the two bisecting lines meet, as a centre, with a radius equal to the distance of D from any one of the sides, describe a circle, and it will be the circle required.

Draw the perpendicular DE, DF, DG. The angles at A are equal by construction, and the angles $A E D$ and $A F D$ are each right angles; therefore the triangles $A D E$ and $A F D$ are equiangular (I. 35), and the side $A D$ is common; therefore the triangles are equal (I. 41), and $D E = D F$. In like manner $D E = D G$. Therefore the circle described from D as a centre with the radius $D E$ will pass through the points F and G; and since the angles at E, F, G are right angles, the sides of the triangle $A B C$ are tangents; therefore the circle EFG is inscribed in the triangle ABC (III. 20).

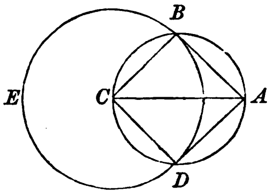
17. *Scholium.* The lines bisecting the angles of a triangle all meet in the same point.

PROBLEM XIV.

18. *Through a given point to draw a tangent to a given circumference.*

1st. If the given point is in the circumference.

Erect a perpendicular to the radius at the given point (3).



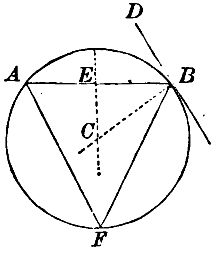
2d. If the given point is without the circumference.

Join the given point A with the centre C of the given circle BDE; on AC as a diameter describe a circle cutting the given circle in B and D. Draw AB and AD, and each will be tangent to the given circle through the given point. For drawing the radii CB, CD, the angles B, D are each right angles (III. 23); therefore AB, AD are tangents to the given circle.

19. *Corollary.* The tangents AB, AD are equal (I. 50).

PROBLEM XV.

20. *Upon a given straight line to describe a segment of a circle which shall contain a given angle.*



Let AB be the given straight line.

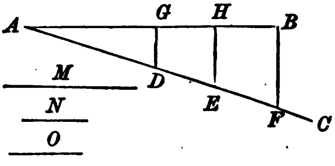
At B make the angle ABD equal to the given angle (5). Draw BC perpendicular to DB; bisect AB in E, and from E draw EC perpendicular to AB. From C, the point of intersection of BC and EC, with a radius equal to CB, describe a circle $AGBF$; BFA is the segment required.

As BD is perpendicular to the radius CB at B, it is a tangent to the circle, and hence the angle ABD is measured by half the arc AGB (III. 54); and any angle BFA inscribed in the segment BFA is also measured by half the arc AGB (III. 21), and is therefore equal to the angle ABD or the given angle.

21. *Corollary.* If the given angle is a right angle, the required segment would be a semicircle described on the given line as a diameter.

PROBLEM XVI.

22. *To divide a given line into parts proportional to given lines.*



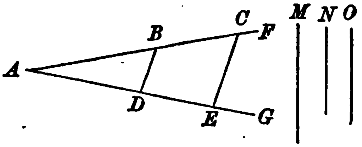
Let it be required to divide AB into parts proportional to M, N, O.

Draw at any angle with AB an indefinite line AC. From A cut off AD, DE, EF equal respectively to M, N, O. Join B to F, and through D and E draw lines parallel to BF. These parallels divide the line as required (II. 16).

23. *Corollary.* By taking M, N, O equal, the given line can be divided into equal parts.

PROBLEM XVII.

24. *To find a fourth proportional to three given lines.*



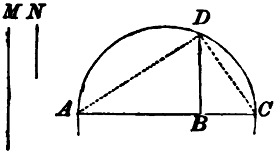
Let it be required to find a fourth proportional to M, N, O.

Draw at any angle with each other the indefinite lines AF, AG. From AF cut off $AB = M$, $BC = N$, and from AG cut off $AD = O$. Join BD and through C draw CE parallel to BD; then DE is the required fourth proportional (II. 16).

25. *Corollary.* By taking AB equal to M, and AD and BC each equal to N, a third proportional can be found to M and N.

PROBLEM XVIII.

26. *To find a mean proportional between two given lines.*



Let it be required to find a mean proportional between M and N.

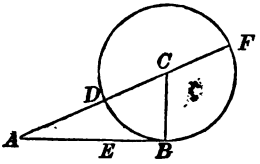
From an indefinite line cut off $AB = M$, $BC = N$; on AC as a diameter describe a semicircle, and at B draw BD perpendicular to AC. BD is the mean proportional required. Join AD, DC. (III. 23; II. 26.)

27. *Definition.* When a line is divided so that one segment is a mean proportional between the whole line and the other segment, it is said to be divided *in extreme and mean ratio.*

PROBLEM XIX.

28. *To divide a given line in extreme and mean ratio.*

Let it be required to divide AB in extreme and mean ratio.



At B draw the perpendicular $BC = \frac{1}{2} AB$; join AC; cut off $CD = CB$, $AE = AD$, and AB is divided at E in extreme and mean ratio.

For, describe a circle with the centre C and radius CB and produce AC to meet the circumference in F; then AF is a secant and AB a tangent of the circle DFB, and therefore (III. 64)

$$AF : AB = AB : AD$$

and (Pn. 18)

$$AF - AB : AB = AB - AD : AD$$

But $$AB = 2\,CB = DF$$

therefore $$AF - AB = AF - DF = AD = AE$$

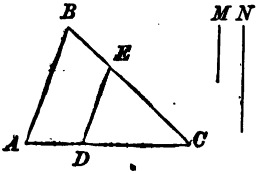
and the proportion becomes

$$AE : AB = EB : AE$$

or (Pn. 16) $$AB : AE = AE : EB$$

PROBLEM XX.

29. *Through a given point in a given angle to draw a line so that the segments included between the point and the sides of the angle may be in a given ratio.*



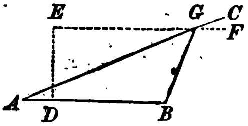
Let it be required to draw through the point D within the angle B a line so that $AD : DC = M : N$.

Draw DE parallel to AB.

Find EC a fourth proportional to M, N, and BE (24); join C to D, and produce CD to A, and AC is the line required (II. 16).

PROBLEM XXI.

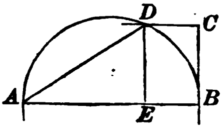
30. *The base, an adjacent angle, and the altitude of a triangle given, to construct the triangle.*



At A of the base AB draw an indefinite line AC making the angle A equal to the given angle; at any point in AB, as D, draw the perpendicular DE equal to the given altitude; through E draw EF parallel to AB cutting AC in G; join GB, and AGB is the triangle required.

PROBLEM XXII.

31. *To construct a parallelogram, having the sum of its base and altitude given, which shall be equivalent to a given square.*

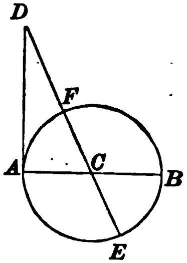


On $A\,B$, the given sum, as a diameter, describe a semicircumference. At any point, as B, in $A\,B$ draw the perpendicular $B\,C$ equal to a side of the given square; through C draw $C\,D$ parallel to $A\,B$, cutting the circumference in D; draw $D\,E$ perpendicular to $A\,B$. $A\,E$, $E\,B$ are one the base and the other the altitude of the parallelogram required (26).

32. *Scholium.* If the side of the square is greater than half the sum of the base and altitude, the construction is impossible.

PROBLEM XXIII.

33. *To construct a parallelogram having the difference between its base and altitude given, which shall be equivalent to a given square.*



On $A\,B$ the given difference, as a diameter, describe a semicircumference. At A draw the perpendicular $A\,D$ equal to a side of the given square; join D with the centre C, and produce $D\,C$ to E. $D\,F$, $D\,E$ are one the base and the other the altitude of the parallelogram required (III. 64).

PROBLEM XXIV.

34. *To construct a square equivalent to a given parallelogram.*

Find a mean proportional between the altitude and base of the given parallelogram (26), and it will be a side of the required square.

PROBLEM XXV.

35. *To construct a square equivalent to a given triangle.*

Find a mean proportional between the base and half the altitude (26), and it will be a side of the required square.

PROBLEM XXVI.

36. *To construct a square equivalent to a given circle.*

Find a mean proportional between the radius and the semicircumference, and it will be a side of the required square.

PROBLEM XXVII.

37. *To construct a square equivalent to the sum of two given squares.*

Construct a right triangle (9) with the sides adjacent to the right angle equal respectively to the sides of the given squares; the hypothenuse will be a side of the required square (II. 27).

38. *Scholium.* By continuing the same process we can find a square equivalent to the sum of any number of given squares.

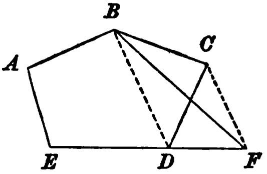
PROBLEM XXVIII.

39. *To construct a square equivalent to the difference of two given squares.*

Construct a right triangle (11), taking as the hypothenuse a side of the greater square, and for one of the sides adjacent to the right angle a side of the other square; the third side of the triangle will be a side of the required square (II. 28).

PROBLEM XXIX.

40. *To construct a triangle equivalent to a given polygon.*



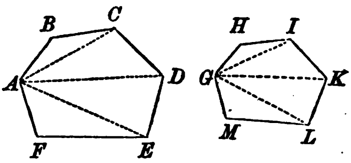
Let AD be the polygon.

Draw BD cutting off the triangle BCD; through C draw CF parallel to BD; join BF, and a polygon $ABFE$ is formed with one side less than the given polygon and equivalent to it. For the triangles BCD and BFD, having the same base BD, and the same altitude, are equivalent; adding to each the common part $ABDE$, we have $ABCDE$ equivalent to $ABFE$. In like manner a polygon with one side less can be found equivalent to $ABFE$, and by continuing the process the sides may be reduced to three, and a triangle obtained equivalent to the given polygon.

41. *Scholium.* Since by (35) a square can be found equivalent to a given triangle, by (40) and (35) a square can be found equivalent to any polygon.

PROBLEM XXX.

42. *On a given line to construct a polygon similar to a given polygon.*



Let AD be the given polygon and ML the given line.

Draw the diagonals AE, AD, AC. At M and L make the angles GML and GLM equal respectively to AFE and AEF, and a triangle GLM will be formed similar to AEF. In like manner on GL construct a triangle similar to ADE; on GK one similar to ACD; on GI one similar to ABC; and the polygons AD,

KG, being composed of the same number of similar triangles similarly situated, are similar (II. 75).

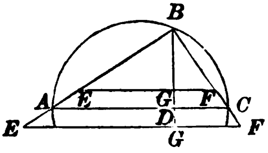
PROBLEM XXXI.

43. *Two similar polygons being given, to construct a similar polygon equivalent to their sum, or to their difference.*

Find a line whose square shall be equivalent to the sum (37), or to the difference (39), of the squares of any two homologous sides of the given polygons, and this will be the homologous side of the required polygon (II. 31). On this line construct (42) a polygon similar to the given polygons.

PROBLEM XXXII.

44. *To construct a square which shall be to a given square in a given ratio.*



On any line AC, as a diameter, describe a semicircumference ABC; divide the line AC at the point D so that $AD : DC$ in the given ratio. Perpendicular to AC draw DB meeting the circumference at B; join BA, BC, and on BC, produced if necessary, take $BF =$ a side of the given square. Through F draw EF parallel to AC, meeting BA in E, and BE is a side of the required square.

For as B is a right angle (III. 23), we have (II. 72)

$$BE^2 : BF^2 = EG : GF$$

But as EF is parallel to AC, we have (II. 47)

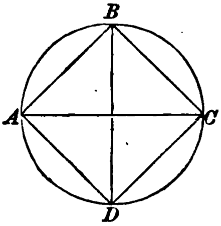
$$EG : GF = AD : DC$$

therefore (Pn. 11)

$$BE^2 : BF^2 = AD : DC$$

PROBLEM XXXIII.

45. *To inscribe a square in a given circle.*

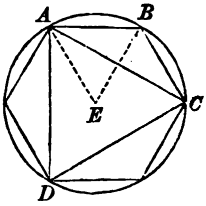


Draw two diameters AC, BD at right angles to each other, and join AB, BC, CD, DA; $ABCD$ is the required square (III. 23; III. 12).

46. *Corollary.* By bisecting the arcs AB, BC, CD, DA; and drawing the chords of these smaller arcs, a regular octagon will be inscribed in the circle. By continuing this bisection regular polygons can be inscribed having the number of their sides 16, 32, 64, and so on.

PROBLEM XXXIV.

47. *To inscribe a regular hexagon in a given circle.*



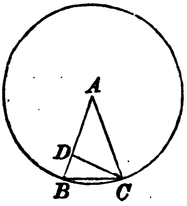
Take AB equal to the radius of the given circle, and it will be a side of the hexagon required (III. 33).

48. *Corollary.* By drawing AC, CD, DA an equilateral triangle will be described in the circle. By bisecting the arcs AB, BC, &c., and continuing this besection as in (46), and drawing the chords of these smaller arcs, regular polygons can be inscribed having the number of their sides 12, 24, 48, 96, and so on.

PROBLEM XXXV.

49. *To inscribe a regular decagon in a given circle.*

Divide the radius AB in extreme and mean ratio at the point D (28), and take $BC = AD$, the greater segment, and it will be the side of the required decagon.



Draw AC, CD. The triangles ACB, DCB are similar (II. 23); for they have the angle B common, and by construction

$$AB : AD = AD : DB$$

but $$AD = BC$$

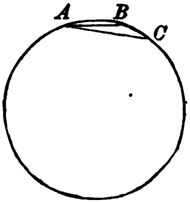
therefore $$AB : BC = BC : BD$$

Therefore, as ACB is isosceles, DCB is also isosceles, and $CD = CB$; therefore also $CD = DA$, and ACD is an isosceles triangle, and the angle $A = ACD$. But the exterior angle $BDC = A + ACD =$ twice the angle A. Therefore, as $B = BDC$, $B =$ twice the angle A. But $B = ACB$; therefore the sum of the three angles A, B, and ACB is equal to five times the angle A; or the angle A is one fifth of two right angles, or one tenth of four right angles; therefore the arc BC is one tenth of the circumference, and the chord BC a side of a regular decagon inscribed in the circle.

50. *Corollary.* By drawing chords joining the alternate angles a regular pentagon will be inscribed. By proceeding as in (46) regular polygons can be inscribed having the number of their sides 20, 40, 80, and so on.

PROBLEM XXXVI.

51. *To inscribe a regular polygon of fifteen sides in a given circle.*



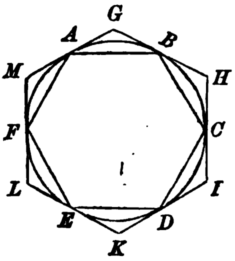
Find by (47) the arc AC equal to a sixth of the circumference, and by (49) the arc AB equal to a tenth of the circumference, and the chord BC will be a side of the polygon required.

For $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$.

52. *Corollary.* Proceeding as in (46) regular polygons can be inscribed having the number of their sides 30, 60, and so on.

PROBLEM XXXVII.

53. *To circumscribe about a given circle a polygon similar to a given inscribed regular polygon.*



Let $A\,D$ be the given inscribed polygon. Through the points A, B, C, D, E, F draw tangents to the circumference. These tangents intersecting will form the polygon required.

For the triangles $A\,G\,B$, $B\,H\,C$, &c. are isosceles (19); and as the arcs $A\,B$, $B\,C$, &c. are equal, the angles $G\,A\,B$, $G\,B\,A$, $H\,B\,C$, $H\,C\,B$, &c. are equal (III. 54); therefore, as the bases $A\,B$, $B\,C$, &c. are equal, these isosceles triangles are equal. Hence the angles G, H, I, K, L, M are equal, and the polygon $M\,I$ is equiangular; and as $GB = BH = HC = CI$, &c., $GH = HI$, &c.; therefore the polygon $M\,I$ is equilateral and regular (II. 32). It is also similar to $A\,D$ (II. 33); and as its sides are tangents it is circumscribed about the circle.

54. *Corollary.* As (45 – 52) regular polygons can be inscribed having the number of their sides 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 32, 40, 48, 60, 64, 80, 96, and so on, regular polygons having the number of their sides represented by these numbers can also be circumscribed about a given circle.

EXERCISES.

55. From two given points to draw two equal lines meeting in a given straight line. (I. 53.)

56. Through a given point to draw a line at equal distances from two other given points.

57. From a given point out of a straight line to draw a line making a given angle with that line. (I. 17.)

58. From two given points on the same side of a given line to draw two lines meeting in the first line and making equal angles with it.

59. From a given point to draw a line making equal angles with the sides of a given angle.

60. Through a given point to draw a line so that the parts of the line intercepted between this point and perpendiculars from two other given points shall be equal.

If the three points are in a straight line, the parts equal what?

61. From a point without two given lines to draw a line such that the part between the two lines shall be equal to the part between the given point and the nearer line.

When is the Problem impossible?

62. To trisect a right angle.

63. On a given base to construct an isosceles triangle having each of the angles at the base double the third angle.

64. To construct an isosceles triangle when there are given

1st. The base and opposite angle.
2d. The base and an adjacent angle.
3d. A side and an opposite angle.
4th. A side and the angle opposite the base.

65. The base, opposite angle, and the altitude given, to construct the triangle. (III. 22.) (20.)

When is the Problem impossible?

66. The base, an angle at the base, and the sum of the sides given, to construct the triangle.

When is the Problem impossible?

67. The base, an angle at the base, and the difference of the sides given, to construct the triangle.

1st. When the given angle is adjacent to the shorter side.
2d. When the given angle is adjacent to the longer side.

When is the Problem impossible?

68. The base, the difference of the sides, and the difference of the angles at the base given, to construct the triangle.

69. The base, the angle at the vertex, and the sum of the sides given, to construct the triangle.

When is the Problem impossible?

70. The base, the angle at the vertex, and the difference of the sides given, to construct the triangle.

71. On a given base to construct a triangle equivalent to a given triangle.

72. With a given altitude to construct a triangle equivalent to a given triangle.

73. Two sides of a triangle and the perpendicular to one of them from the opposite vertex given, to construct the triangle.

74. Two of the perpendiculars from the vertices to the opposite sides and a side given, to construct the triangle.

1st. When one of the perpendiculars falls on the given side.

2d. When neither of the perpendiculars falls on the given side.

75. An angle and two of the perpendiculars from the vertices to the opposite sides given, to construct the triangle.

1st. When one of the perpendiculars falls from the vertex of the given angle.

2d. When neither of the perpendiculars falls from the vertex of the given angle.

76. An angle and the segments of the opposite side made by a perpendicular from the vertex given, to construct the triangle.

77. Given an angle, the opposite side, and the line from the given vertex to the middle of the given side, to construct the triangle.

When is the Problem impossible?

78. An angle, a perpendicular from another angle to the opposite side, and the radius of the circumscribed circle given, to construct the triangle.

When is the Problem impossible?

79. To divide a triangle into two parts in a given ratio,

1st. By a line drawn from a given point in one of its sides.

2d. By a line parallel to the base.

80. To trisect a triangle by straight lines drawn from a point within to the vertices.

81. Parallel to the base of a triangle to draw a line equal to the sum of the lower segments of the two sides.

82. Parallel to the base of a triangle to draw a line equal to the difference of the lower segments of the two sides.

83. To inscribe in a given triangle a quadrilateral similar to a given quadrilateral.

84. To divide a given line so that the sum of the squares of the parts shall be equivalent to a given square.

85. To construct a parallelogram when there are given,

1st. Two adjacent sides and a diagonal.
2d. A side and two diagonals.
3d. The two diagonals and the angle between them.
4th. The perimeter, a side, and an angle.

86. To construct a square when the diagonal is given.

87. To construct a parallelogram equivalent to a given triangle and having a given angle.

88. To draw a quadrilateral, the order and magnitude of all the sides and one angle given.

Show that sometimes there may be two different polygons satisfying the conditions.

89. To draw a quadrilateral, the order and magnitude of three sides and two angles given.

1st. The given angles included by the given sides.
2d. The two angles adjacent, and one adjacent to the unknown side.
3d. The two angles being opposite each other.
4th. The two angles being both adjacent to the unknown side.
In any of these cases can more than one quadrilateral be drawn?

90. To draw a quadrilateral, the order and magnitude of two sides and three angles given.

1st. The given sides being adjacent.
2d. The given sides not being adjacent.

91. In a given circle to inscribe a triangle similar to a given triangle.

92. Through a given point to draw to a given circle a secant such that the part within the circle may be equal to a given line.

93. With a given radius to draw a circumference,

1st. Through two given points.

2d. Through a given point and tangent to a given line.

3d. Through a given point and tangent to a given circumference.

4th. Tangent to two given straight lines.

5th. Tangent to a given straight line and to a given circumference.

6th. Tangent to two given circumferences.

State in each of these cases how many circles can be drawn, and when the construction is impossible.

94. To draw a circumference,

1st. Through two given points and with its centre in a given line.

2d. Through a given point and tangent to a given line at a given point.

3d. Tangent to a given line at a given point, and also tangent to a second given line.

4th. Tangent to three given lines.

5th. Through two given points and tangent to a given line.

6th. Through a given point and tangent to two given lines.

95. To draw a tangent to two circumferences.

There can be drawn,

1st. When the circles are external to each other, four tangents.

2d. When the circles touch externally, three.

3d. When the circles cut, two.

4th. When the circles touch internally, one.

5th. When one circle is within the other, none.

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